Кафедра електроніки, робототехніки і хнологій моніторингу та інтернету речей

Факультет аеронавігації, електроніки та телекомунікацій (ФАЕТ)

Електронні системи

Electronic Systems

Lecture #13

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Орієнтовний тематичний план лекцій

Основи теорії систем, сигнали і первинні перетворювачі електронних систем

1.Вступ. Визначення і термінологія, класифікація	2	
2.Характеристики електронних систем	2	
3.Теорія систем, аналіз електронних систем	2	
4.Первинні перетворювачі електронних систем	4	
5.Сигнали електронних систем	2	
6.Компоненти і обробка сигналів в ЕС	1	**7 семест**
7.Експлуатаційні характеристики електронних систем	2	
8.Технічні характеристики електронних систем	2	
9.Технічна реалізація системи	1	
10.Електронні системи локації	18	
11. Електронні системи зв'язку	8	**8 семест**
12. Електронні системи авіоніки	19	
Всього годин	**63**	

Електронні системи локації

1. Основні терміни, принцип дії, класифікація та застосування. 2
2. Відбиваючі властивості об'єктів. 2
3. **Виявлення сигналів. 4**
4. Дальність дії локаційної системи. 2
5. Роздільна здатність локаційної системи. 2
6. Вимірювання дальності та швидкості об'єктів. 2
7. Вимірювання кутових координат. 2
8. Методи підвищення роздільної здатності і точності вимірювань. 2

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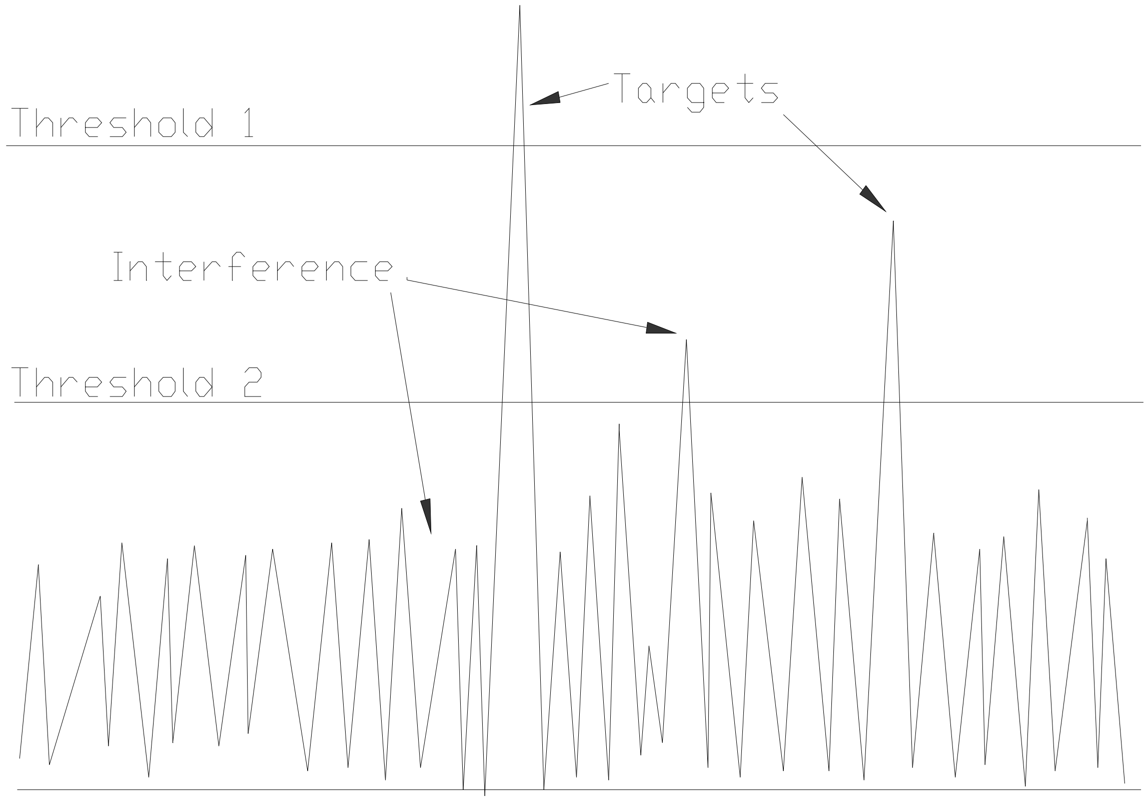
Signal detection

Що таке виявлення?

- Ми давали визначення.
- Бінарне виявлення.
- Прийняття рішення.
- Іспит на поріг.

Threshold Detection – Порогове виявлення

$$x(t) = s(t) + n(t)$$



Models of signals and interferences

$$x(t) = s(t) + n(t)$$

Models of noise

- White noise
- White noise with limited spectrum, or quasi-white noise (QWN)
- QWN with zero carrier
- QWN with carrier

Models of signals and interferences

Models of signals

1. Signal with known parameters – берем просто відомий сигнал
2. Signal with unknown initial phase
3. Signal with random initial phase and amplitude
4. Burst of pulses with constant amplitudes and random initial phases of each pulse (non-coherent burst)
5. Coherent burst
6. Burst of pulses with fluctuated envelope and random initial phases:
   6.1 Amplitudes fluctuate independently (fast fluctuations)
   6.2 Amplitudes are dependent between themselves (slow fluctuations)

Swerling 1 & 3　　　　Swerling 2 & 4

Quality characteristics of radar detection

- The **decision** must be made at two mutually exclusive conditions :

  A_1 – Target is present $\Rightarrow x(t)=n(t)+s(t)$

  A_0 – Target is absent $\Rightarrow x(t)=n(t)$

- The detector must accept one of two hypotheses:

  A^*_1 – Target is present $\Rightarrow \mathrm{s(t)}\neq 0$

  A^*_0 – Target is absent $\Rightarrow \mathrm{s(t)}=0$

Можливі 4 ситуації суміщення випадкових подій «рішення» і «умови» ("decisions" and "conditions"):

Correct detection (Detection)	$A^*_1 A_1$
Missing target (Missing)	$A^*_0 A_1$
False Alarm	$A^*_1 A_0$
Correct undetection	$A^*_0 A_0$

Some cost can be put in correspondence to every erroneou decision:

C_{10} – cost of FA; C_{01} – cost of Missing

$$C_{11} = C_{00} = 0$$

Then, a detection system can be characterized by the **mean cost of erroneous decisions** (mathematical expectation)

$$\bar{r} = \cancel{C_{11}P(A_1^* A_1)} + C_{01}P(A_0^* A_1) + C_{10}P(A_1^* A_0) + \cancel{C_{00}P(A_0^* A_0)}$$

$$\bar{r} = C_{01}P(A_0^* A_1) + C_{10}P(A_1^* A_0)$$

Quality of detection is higher, if $\bar{r}$ is smaller

Тобто оптимальною можна вважати таку обробку, за якої $\bar{r} = \min$

From the probability theory it is known:

Joint probability of two events is equal to the probability of one event multiplied by conditional probability of other eve
under the condition that the first event has been occurred.

Mean cost of erroneous decisions = Average Risk

$$\bar{r} = C_{01} P(A_1) P(A_0^* / A_1) + C_{10} P(A_0) P(A_1^* / A_0)$$

f a radar detector minimizes the Average Risk, such radar detector is optimal **Average Risk = min**

$$P(A_0^* / A_1) = \bar{D} = 1 - D$$

Імовірність пропуску цілі

$$P(A_1^* / A_0) = F$$

Імовірність хибної тривоги

Likelihood ratio derivation

We have derived the expression for mean cost of error as:

$$\bar{r} = \underbrace{C_{10}P(A_0)}_{a} \cdot F + \underbrace{C_{01}P(A_1)}_{b} \cdot (1 - D)$$

$$\bar{r} = aF + b(1 - D)$$

Divide both parts by b :

$$R = \frac{\bar{r}}{b}$$

$$R = \frac{a}{b}F + (1 - D)$$

$$\boxed{\frac{a}{b} = l_0 = \frac{C_{10}P(A_0)}{C_{01}P(A_1)}}$$

$$R = (1 - D) + l_0 F$$

$$R = 1 - (D - l_0 F)$$

From this, it is clear that minimum R will be achieved when the difference

$(D - l_0 F)$ is maximum

$(D - l_0 F) = \max$ - Weighting criterion

$$(D - l_0 F) = \max$$

- The last expression shows that based on the set of requirements – increasing conditional probability D and decreasing conditional probability F one should tend to increasing the weighted difference $D - l_0 F$
- The multiplier l_0 is named weighted multiplier. It depends on the ratio of error costs (every kind of errors) and a priori probabilities…
- Now we can build a decision rule on the basis of weighting criterion.

Transition to multidimensional (discrete) sample

$x(t)$ A realization of continuous function

$\{x_1, x_2, \dots x_n\}$ Multidimensional random values

$\{t_1, t_2, \dots t_n\}$ in points of time

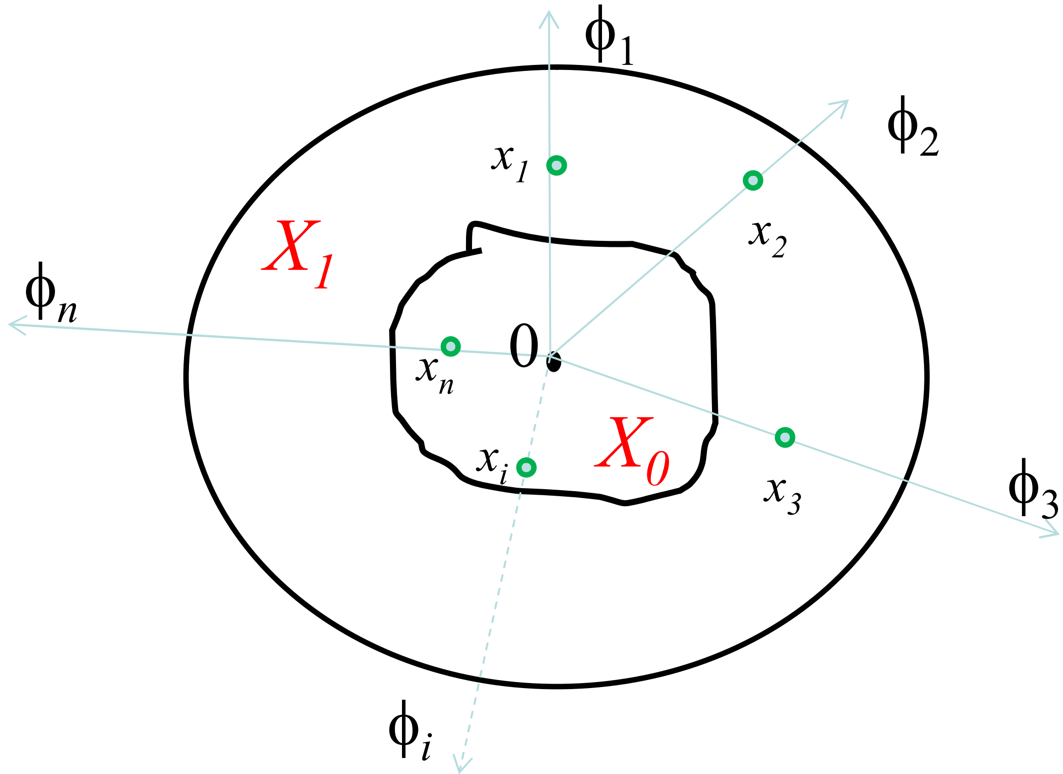
$[0, \quad T]$ on time interval, T - duration of signal $x(t)$

So, we work with a sample $x_1, x_2, \dots, x_n$

We must make one of two mutually exclusive decisions:

A^*_1 or A^*_0

The decision rule must divide n-dimensional space of samples onto two subspaces X_1 and X_0. Subspaces X_1 and X_0 are adjacent but non-overlaping.



Sample

$$X = x_1, x_2, \dots, x_n$$

$$X \in X_1 \Rightarrow A_1^*$$

$$X \in X_0 \Rightarrow A_0^*$$

Likelihood function

- Let us denote the conditional multi-dimensional probability densities of the signal sample $X = x_1, x_2, \ldots, x_n$ as:

$W_{sn}(X)$ - under condition $\boldsymbol{A_1}$

$$x(t)=n(t)+s(t)$$

$W_n(X)$ - under condition $\boldsymbol{A_0}$

$$x(t)=n(t), \text{ that is, } s(t) = 0$$

Such conditional probability density is named Likelihood function

$$\boxed{W_{sn}(X) = W(X / A_1)}$$

$$\boxed{W_n(X) = W(X / A_0)}$$

Likelihood Ratio

$$\frac{W_{sn}(X)}{W_n(X)} \geq \lambda_0$$

$$X = x_1, x_2, \ldots, x_n$$

$$\frac{W_{sn}(X)}{W_n(X)} < \lambda_0$$

$$\boxed{\lambda(X) = \frac{W_{sn}(X)}{W_n(X)}}$$

Ratio of likelihood functions, or Likelihood Ratio

Likelihood function – is a conditional probability density

Solution of statistical problem of synthesizing optimal radar detector

1. Choice and justification of Optimality Criteria (on the basis of the features of a concrete task)
2. Synthesis of Algorithm – finding a math rule of optimal detection problem solution
3. Implementation of the Algorithm using electronics means and software – finding a structural diagram of the detector
4. Investigation of optimal detector characteristics
5. Comparison of optimal and real detectors

Criteria of Optimality

- Average risk criterion
- Ideal observer criterion
- Criterion of minimal weighted error probability
- Neumann- Pierson criterion
- Wald criterion (Sequential Observation)

Conclusion on the first portion

As a result of this part of the lecture we know:

- Models of signals and interferences (noise)
- Quality characteristics of radar detection
- General statement and solution of the problem of detectio
  [$\lambda(x)$>=λo]
- Criteria of Optimality
- Next step is – the synthesis of algorithms and structural diagrams of optimal detectors for specific models of signa

Binary detection of a known signal

$$x(t) = A \cdot s(t) + n(t)$$

$$\begin{cases} A=0 & \text{(situation } A_0\text{)} \\ A=1 & \text{(situation } A_1\text{)} \end{cases}$$

$n(t)$ – Gaussian white noise

$$x_k = n_k + s_k, \qquad k = 1, 2, \dots, N$$

Conditional probability density of k-th readings of input oscillation $x(t)$

$w_n(x_k)$ at A=0 - Gauss with zero mean

$w_{sn}(x_k)$ at A=1- Gauss with mean = s_k

We need $w_n(x_k)$ and $w_{sn}(x_k)$ to calculate the likelihood ratio

In order to find joint multi-dimensional probability densities of a sample

$$X = \{x_1, x_2, \ldots x_n\}$$

we should know statistical relationship of the processes in the points of the sample.

These points are spaced by the intervals

$$k \cdot \Delta t \qquad \Delta t = \frac{1}{2 f_{\max}}$$

Statistical relationship is described by CF, which is equal to zero in the points of the sample (readings).

In other words, the readings in the sample $x_1, x_2, \ldots, x_n$ are statistically independent. That is why the joint multi-dimensional PDF is equal to the product of PDFs of each separate values $x_1, x_2, \ldots, x_n$:

$$W_n(X) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^N \exp\left(-\sum_{k=1}^{N}\frac{x_k^2}{2\sigma^2}\right)$$

A=0, situation A_0

$$W_{sn}(X) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^N \exp\left(-\sum_{k=1}^{N}\frac{(x_k - s_k)^2}{2\sigma^2}\right)$$

A=1, situation A_1

Variance σ^2 in both cases is the same, and $\sigma^2 = N_0 f_{\max} = \dfrac{N_0}{2\Delta t}$

$$\boxed{\lambda(X)=\frac{W_{sn}(X)}{W_n(X)}}$$

$$\lambda(X)=\exp\left[-\sum_{k=1}^{N}\frac{\cancel{x_k^2}-2x_k s_k+s_k^2-\cancel{x_k^2}}{2\sigma^2}\right]$$

$$\lambda(X)=\exp\left[\frac{1}{2\sigma^2}\sum_{k=1}^{N}(2x_k s_k-s_k^2)\right]$$

$$\sigma^2=N_0 f_{\max}=\frac{N_0}{2\Delta t}$$

$$\lambda(X)=\exp\left[\frac{\Delta t}{N_0}\sum_{k=1}^{N}(2x_k s_k-s_k^2)\right]$$

$$\boxed{\lambda(X)=\exp\left[-\frac{1}{N_0}\sum_{k=1}^{N}s_k^2\Delta t+\frac{2}{N_0}\sum_{k=1}^{N}x_k s_k\Delta t\right]}$$

Limiting process to white noise case

$$f_{\max} \to \infty \qquad \Delta t \to 0$$

$$\lim_{\Delta t \to 0} \sum_k s_k^2 \Delta t = \int s^2(t)\,dt = E$$

$$\lim_{\Delta t \to 0} \sum_k x_k s_k \Delta t = \int_{-\infty}^{\infty} s(t)\, x(t)\, dt$$

$$z = \int_{-\infty}^{\infty} s(t)\, x(t)\, dt = \int_{0}^{T} s(t)\, x(t)\, dt$$

Final form of the Likelihood Ratio for known signal detection

$$\lambda[x(t)] = e^{-\frac{E}{N_0}} \cdot e^{-\frac{2z}{N_0}}$$

One can see that λ is a monotonous function of correlation integral z, which can be calculated, using the received realization $x(t)$

Comparison of Likelihood Ratio λ with threshold λ_0 is equivalent to a comparison of cor. int. z with corresponding threshold z_0

Threshold for Correlation Integral

Condition $\lambda(X) \geq l_0$ is equivalent to $\ln \lambda \geq \ln l_0$

$$\lambda = e^{-\frac{E}{N_0}} \cdot e^{-\frac{2z}{N_0}} \qquad \lambda_0 = e^{-\frac{E}{N_0}} \cdot e^{-\frac{2z_0}{N_0}}$$

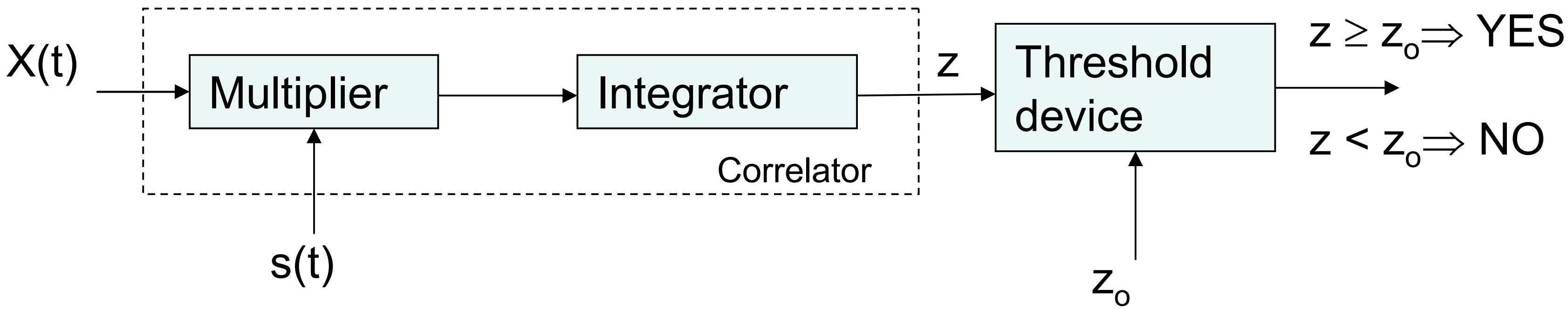
$$\ln \lambda_0 = -\frac{E}{N_0} + \frac{2z_0}{N_0}$$

whence:

$$\boxed{z_0 = \frac{N_0}{2} \ln l_0 + \frac{E}{2}}$$

Synthesis of Optimal Signal Detector

- Likelihood Ratio
- Correlation Integral
- Correlation Receiver
- Comparison with threshold
- Decision Making



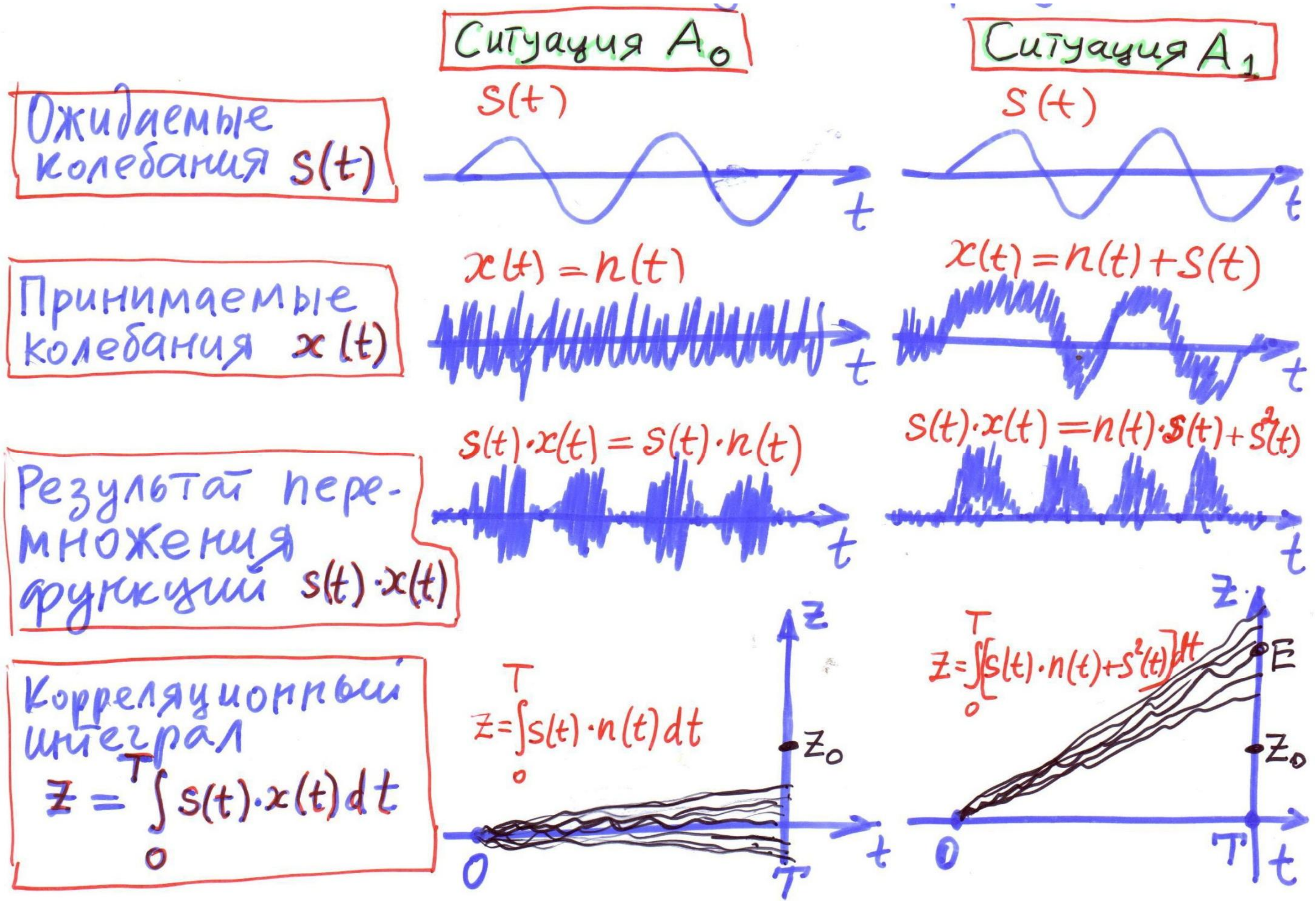
Physical interpretation of correlation processing

Expected oscillations

Received oscillations

Result of multiplication s(t)·x(t)

Correlation integral z



Thus, at the output of the correlator we have:

$$z_n(T) = \int_0^T n(t)s(t)dt$$ in case of noise at the input

$$z_{sn}(T) = \int_0^T [s(t) + n(t)]s(t)dt$$ in case of mixture noise and signal $s(t)$

Important! : z_n and z_{sn} are random values. That is why exceeding (or non-exceeding) the threshold occurs with probabilities less than unity.

Decision making rule:

$$z(T) \geq z_0 \Rightarrow A_1^*$$ Target is present

$$z(T) < z_0 \Rightarrow A_0^*$$ Target is absent

- Probability density distribution of random value z (z_n in case A_0 and z_{sn} in case of A_1) and the value z_0 of the threshold define the probabilities of correct and erroneous decisions
- PDFs $w_n(z)$ and $w_{sn}(z)$

$$w_n(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}}$$

We need to know the variance σ_z^2

$$\sigma_z^2 = M\{z^2\} = \overline{z^2}$$

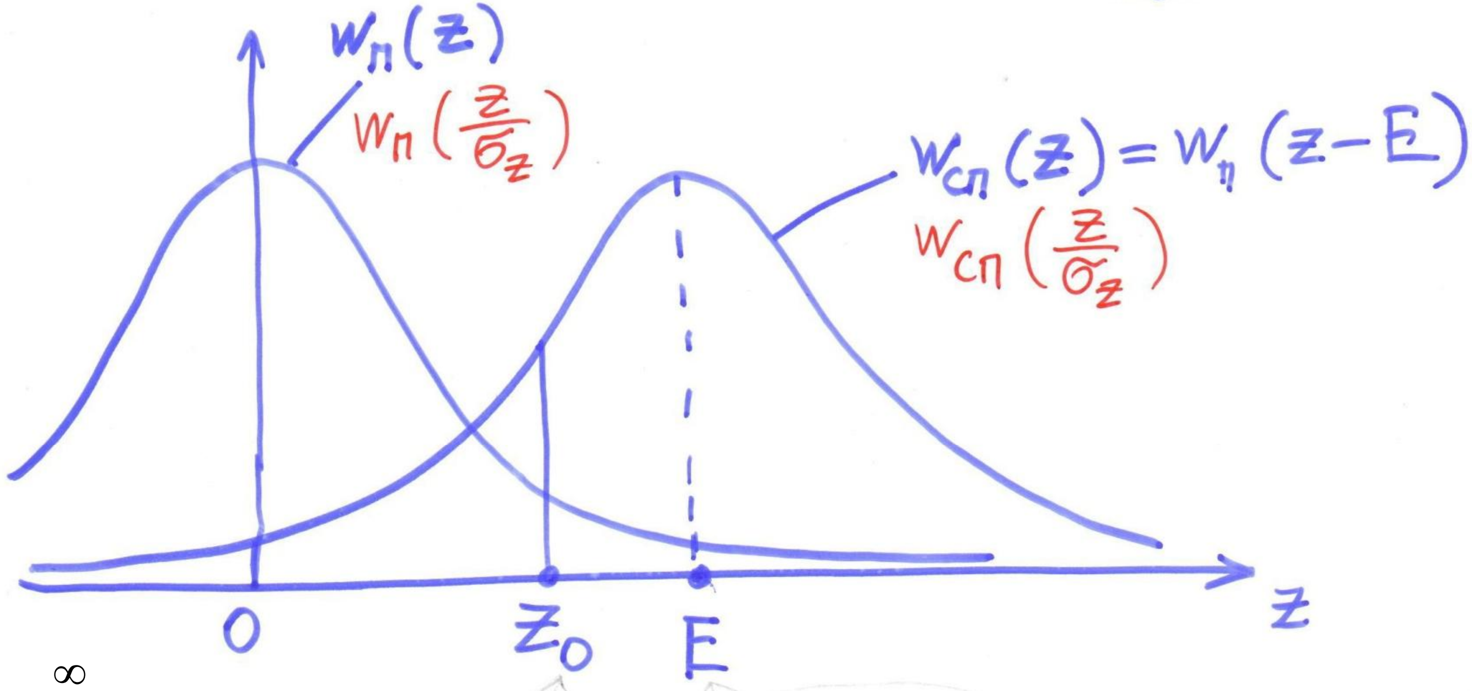
$$z^2 = \int_{-\infty}^{\infty} n(t)s(t)dt \cdot \int_{-\infty}^{\infty} n(t)s(t)dt = \iint_{\infty} s(t)s(t')n(t)n(t')dt'dt$$

$$\overline{z^2} = \iint_{\infty} s(t)s(t') \cdot \overline{n(t)n(t')}dt'dt$$

$$\overline{n(t)n(t')} = \frac{N_0}{2}\delta(t-t') \qquad \int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

$$\sigma_z^2 = \overline{z^2} = \frac{N_0}{2}\int_{-\infty}^{\infty} s^2(t)dt = \frac{N_0 E}{2}$$

This variance σ_z^2, derived by us, defines completely the curve $w_n(z)$



$$E = \int_{-\infty}^{\infty} s^2(t)dt$$

$q = \frac{z_0}{\sigma_z}$ $\quad q = \frac{E}{\sigma_z} = \sqrt{\frac{2E}{N_0}}$

It is convenient to graduate the axis of abscissa in relative values $\boxed{\frac{z}{\sigma_z}}$

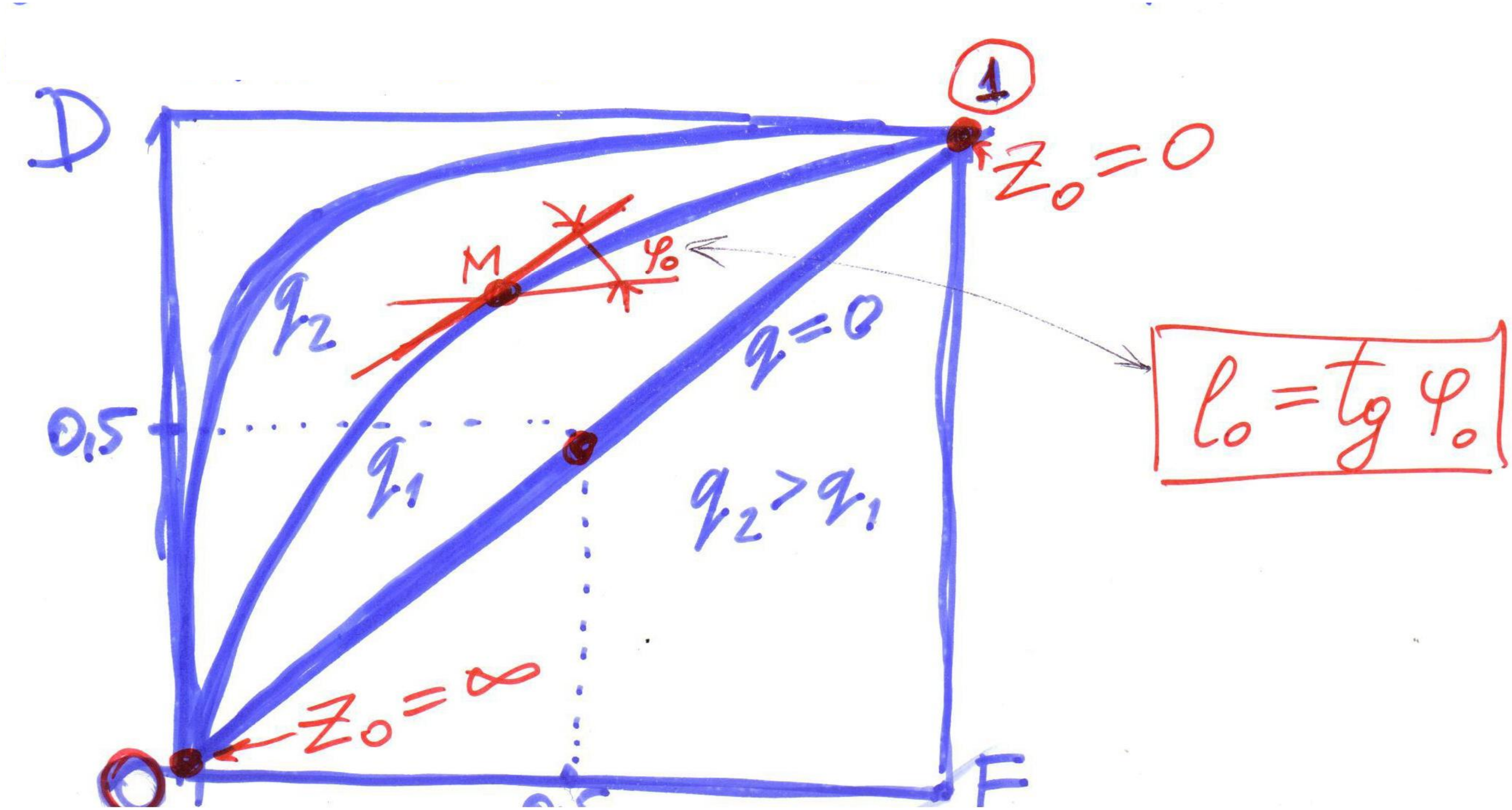
$$q = \frac{E}{\sigma_z} = \sqrt{\frac{2E}{N_0}}$$

- detection parameter equals to SNR on voltage at the output of optimal detector.

Probabilities of Detection and False Alarm

Changing the threshold changes both D & F.

Operating characteristics of a radar detector – is a dependence D of F at given SNR $q^2=2E/N_0$



Probabilities of Detection and False alarm

$$F = \frac{1}{2}\left[1 - \Phi\left(q_0\right)\right]$$

$q_0 \quad - threshold$ (relative threshold level)

$q \quad - signal - to - noise\ ratio$ $\qquad q = \sqrt{\frac{2E}{N_0}}$

$$D = \frac{1}{2}\left[1 - \Phi\left(q_0 - q\right)\right]$$

$$q_0 = \frac{z_0}{\sigma_z} = \frac{\left(\frac{N_0}{2}\ln l_0 + \frac{E}{2}\right)}{\sqrt{\frac{N_0 E}{2}}} = \sqrt{\frac{N_0 E}{2}}\ln l_0 + \sqrt{\frac{N_0 E}{2}}$$

$$\boxed{\sigma_z^2 = \frac{N_0 E}{2}}$$

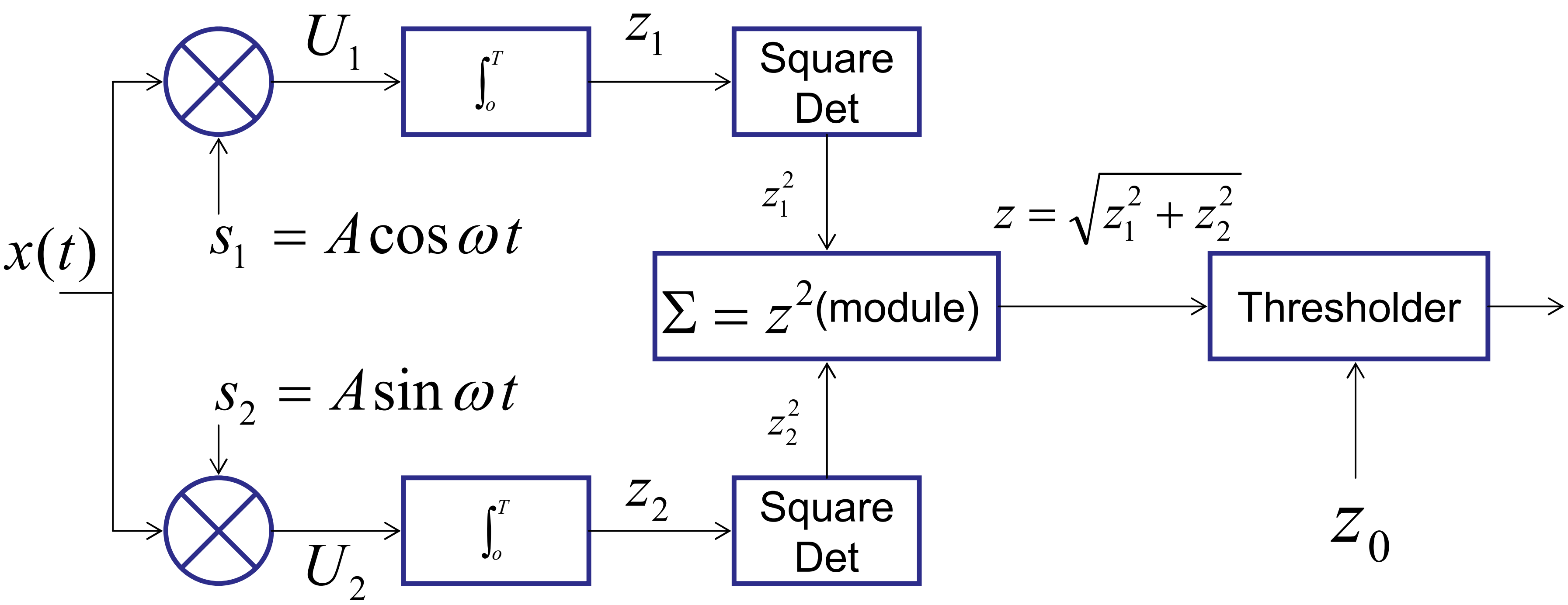
$$\Phi(\xi_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi_0} \exp\left(-\frac{\xi^2}{2}\right)d\xi$$

\- probability integral

Detection of a signal with a random initial phase

- At the output of the multiplier the amplitude will be proportional to the phase difference between phases of a signal $\boldsymbol{x}(\boldsymbol{t})$ and the reference voltage $\boldsymbol{s}(\boldsymbol{t})$, that is, a result of the multiplication is a random value.
- Two parallel channels are used in order to eliminate a randomness of changing output voltage, and the reference voltages in these channels are 90^{o} phase-shifted.

Detection of a signal with a random initial phase



$$U_1 = k_1 \cos\psi_{dif} \qquad z_1 = k_2 \cos\psi_{dif}$$

$$z^2 = z_1^2 + z_2^2 = k_2^2(\cos^2\psi_{dif} + \sin^2\psi_{dif}) = Const$$

$$U_2 = k_1 \sin\psi_{dif} \qquad z_2 = k_2 \sin\psi_{dif}$$

Probabilities of Detection and False Alarm

Situation	PDF	Note
A_0	$w_n(z) = \frac{z}{\sigma_z^2} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$	Rayleigh
A_1	$w_{sn}(z) = \frac{z}{\sigma_z^2} \exp\left(-\frac{z^2 + E^2}{2\sigma_z^2}\right) I_0\left(\frac{zE}{\sigma_z^2}\right)$	Generalized Rayleigh (Rice)

$$F = \int_{z_0}^{\infty} w_n(z)dz = \int_{q_0}^{\infty} w_n(q)dq \qquad D = \int_{z_0}^{\infty} w_{sn}(z)dz = \int_{q_0}^{\infty} w_{sn}(q)dq$$

$$q_0 = \frac{z_0}{\sigma_z} \qquad q = \frac{z}{\sigma_z}$$

Detection of a signal with random initial phase and amplitude

- This case corresponds to Model 3:

$$s(t) = \alpha A(t)\cos[\omega_0 t + \varphi(t) + \varphi_0]$$

- Random quantity α is distributed by Rayleigh law

$$w(\alpha) = \frac{\alpha}{\sigma^2}\exp(-\frac{\alpha^2}{2\sigma^2})$$

- Random quantity φ_0 – uniformly distributed

$$w(\varphi_0) = 1/(2\pi)$$

- Joint PDF (independent random quantities):

$$w(\alpha,\varphi_0) = \frac{1}{2\pi}\cdot\frac{\alpha}{\sigma^2}\exp(-\frac{\alpha^2}{2\sigma^2})$$

- Likelihood radio for known signal

$$\lambda(x) = \exp[-\frac{E}{N_0} + \frac{2z}{N_0}]$$

In order to find Likelihood Ratio for this case we must:

- Calculate energy of the signal as function of random amplitude $E(\alpha)$
- Calculate correlation integral as function of α and φ_0 $z(\alpha, \varphi_0)$
- Substitute these values into Ex. ★

$$\boxed{\lambda(x, \alpha, \varphi_0)}$$

- Average the expression obtained over the random parameters α and φ_0

$$\boxed{\lambda(x) = \iint \lambda(x, \alpha, \varphi_0) w(\alpha) w(\varphi_0) \, d\alpha \, d\varphi_0}$$

Result

- The Likelihood Ratio for the signal with random initial phase and random amplitude is a monotonous function of MODULE value of correlation integral z similarly to the case when only initial phase is unknown.
- The structure of the detector for Model 3 is the same as for Model 2. Only optimal threshold is different.
- Method of calculating Probabilities of Detection and False Alarm is also similar.

Probabilities of Detection and False Alarm

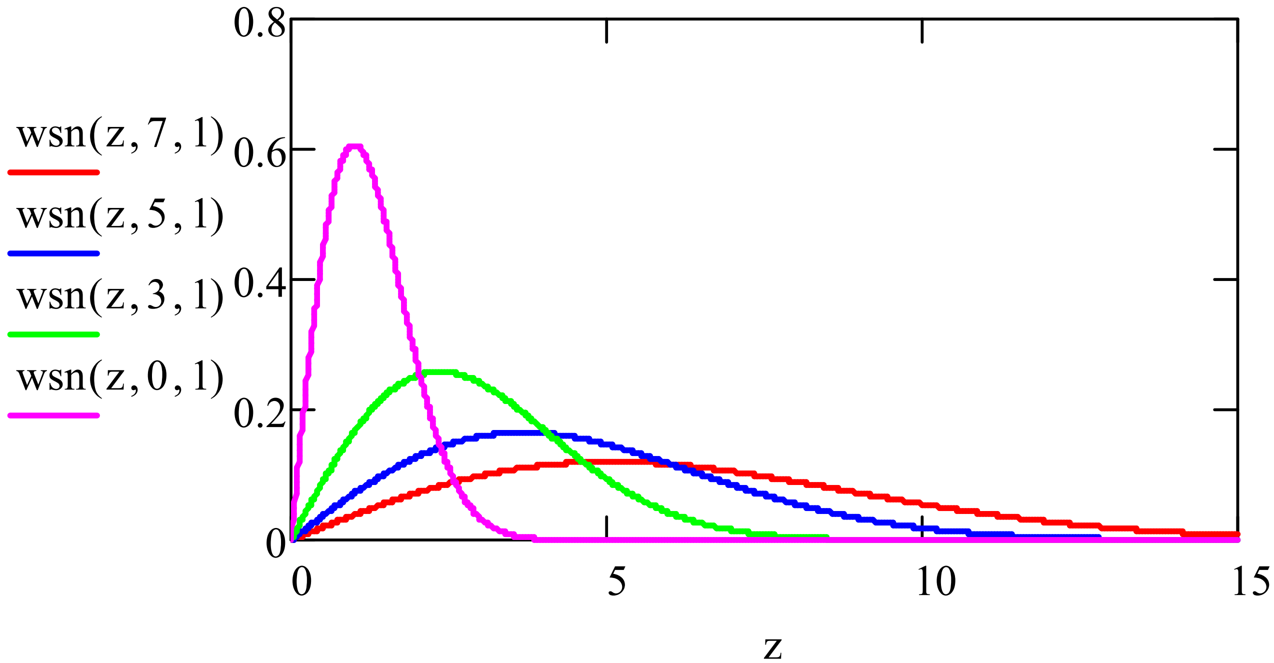
Situation	PDF	Note
A_0	$w_n(z) = \frac{z}{\sigma_z^2} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$	Rayleigh
A_1	$w_{sn}(z) = \frac{2z}{2\sigma_z^2 + E^2} \exp\left(-\frac{z^2}{2\sigma_z^2 + E^2}\right)$	At E=0 - Rayleigh

$$D = \exp\left(-\frac{\frac{q_0^2}{2}}{1 + \frac{q^2}{2}}\right) \qquad F = \exp\left(-\frac{q_0^2}{2}\right)$$

$$q_0 = \frac{z_0}{\sigma_z} \qquad q = \sqrt{\frac{2E}{N_0}}$$

Плотности вероятностей корреляционного интеграла для сигнала со случайным
амплитудуми и начальными фазами.
При E=0 имеем ситуацию A0 - сигнала нет. Тогда получается чистый Релей

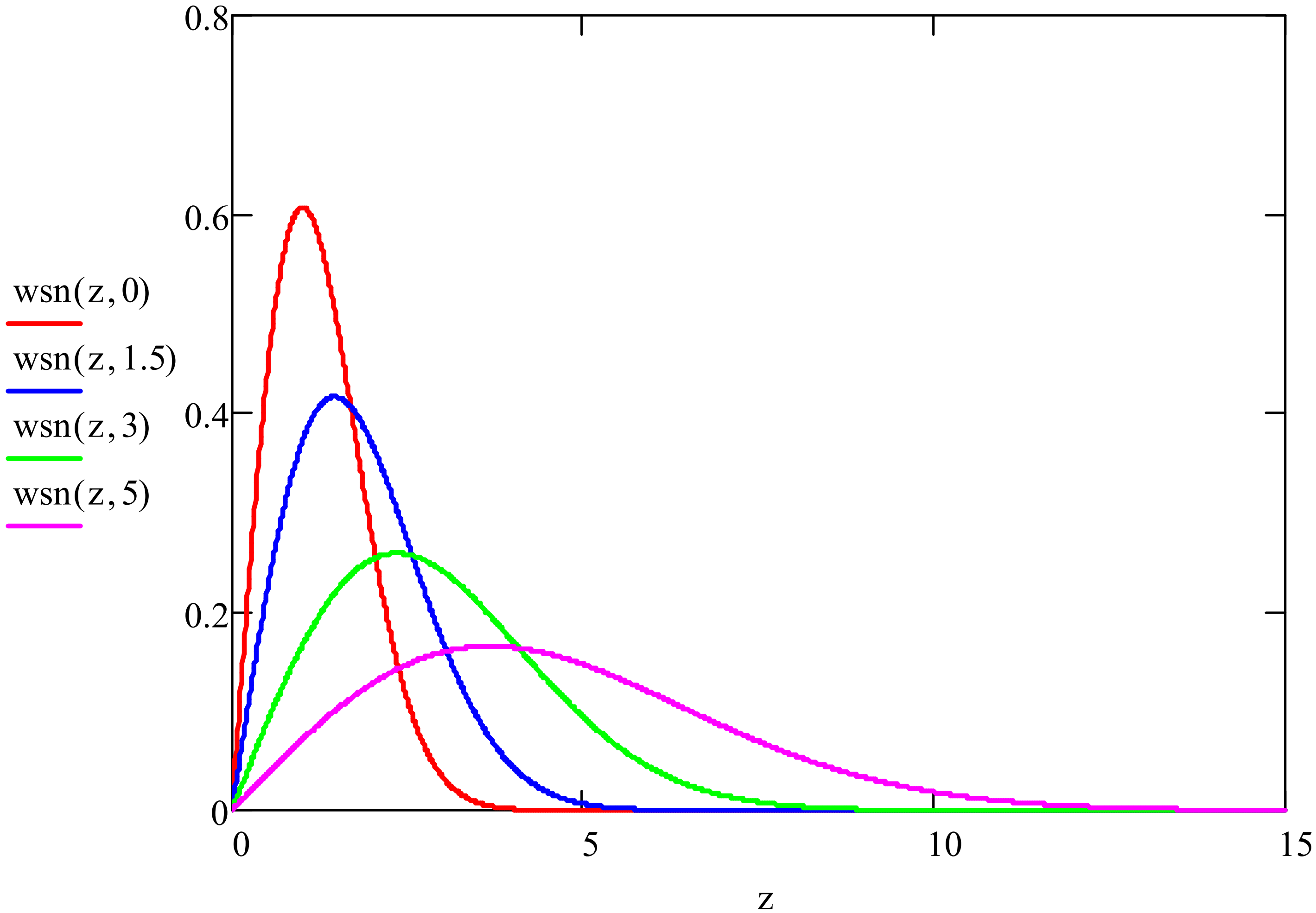
$$wsn(z,E,\sigma) := \frac{2\cdot z}{2\cdot\sigma^2 + E^2}\cdot\exp\left(\frac{-z^2}{2\cdot\sigma^2 + E^2}\right) \qquad z := 0, 0.01 .. 15$$



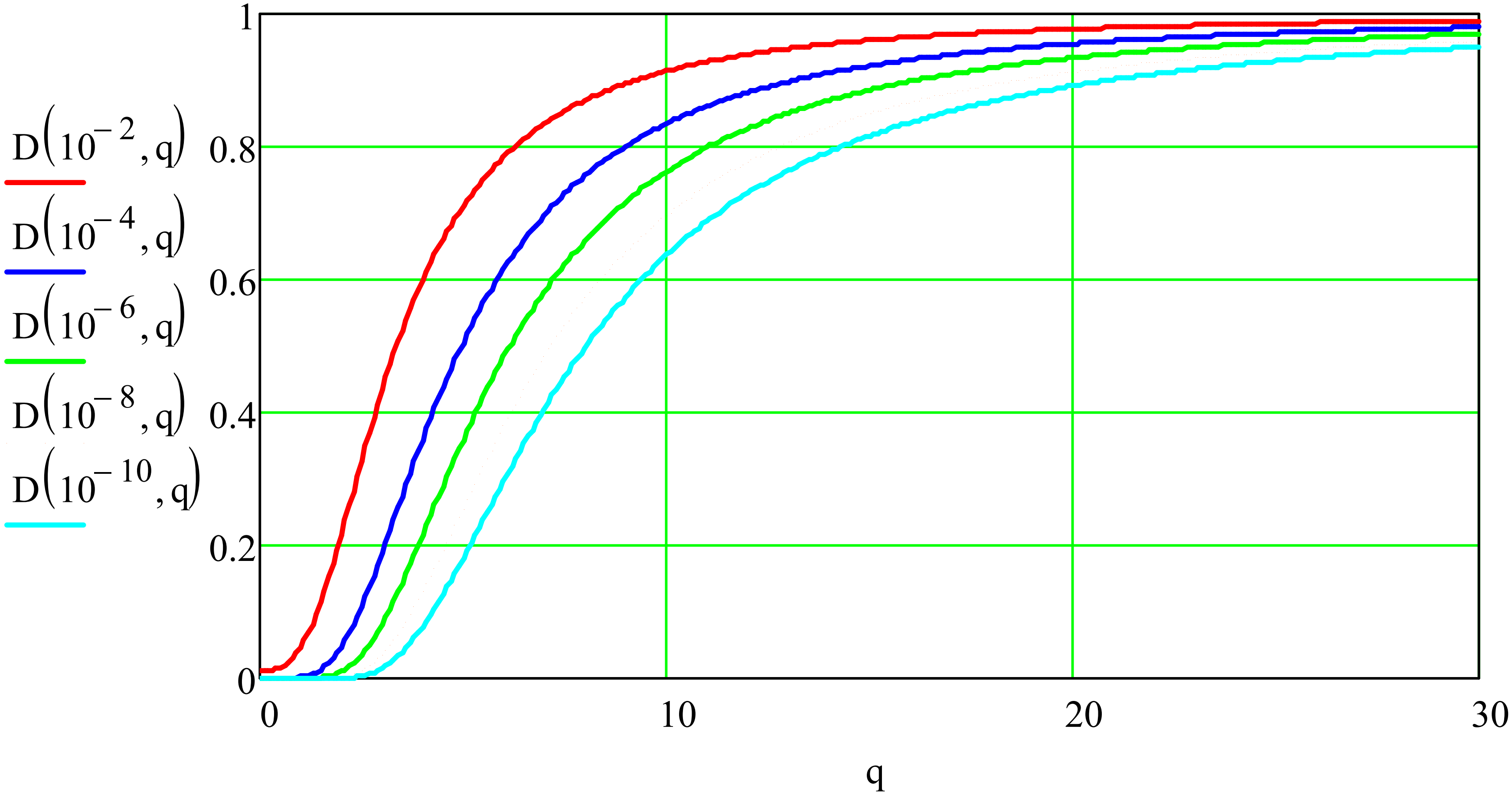
Относительные величины

$$q := \sqrt{\frac{2\cdot E}{No}} \qquad qo := \frac{zo}{\sigma}$$

$$wsn(z,q) := \frac{2\cdot z}{2\cdot 1^2 + q^2}\cdot \exp\left(\frac{-z^2}{2\cdot 1^2 + q^2}\right)$$

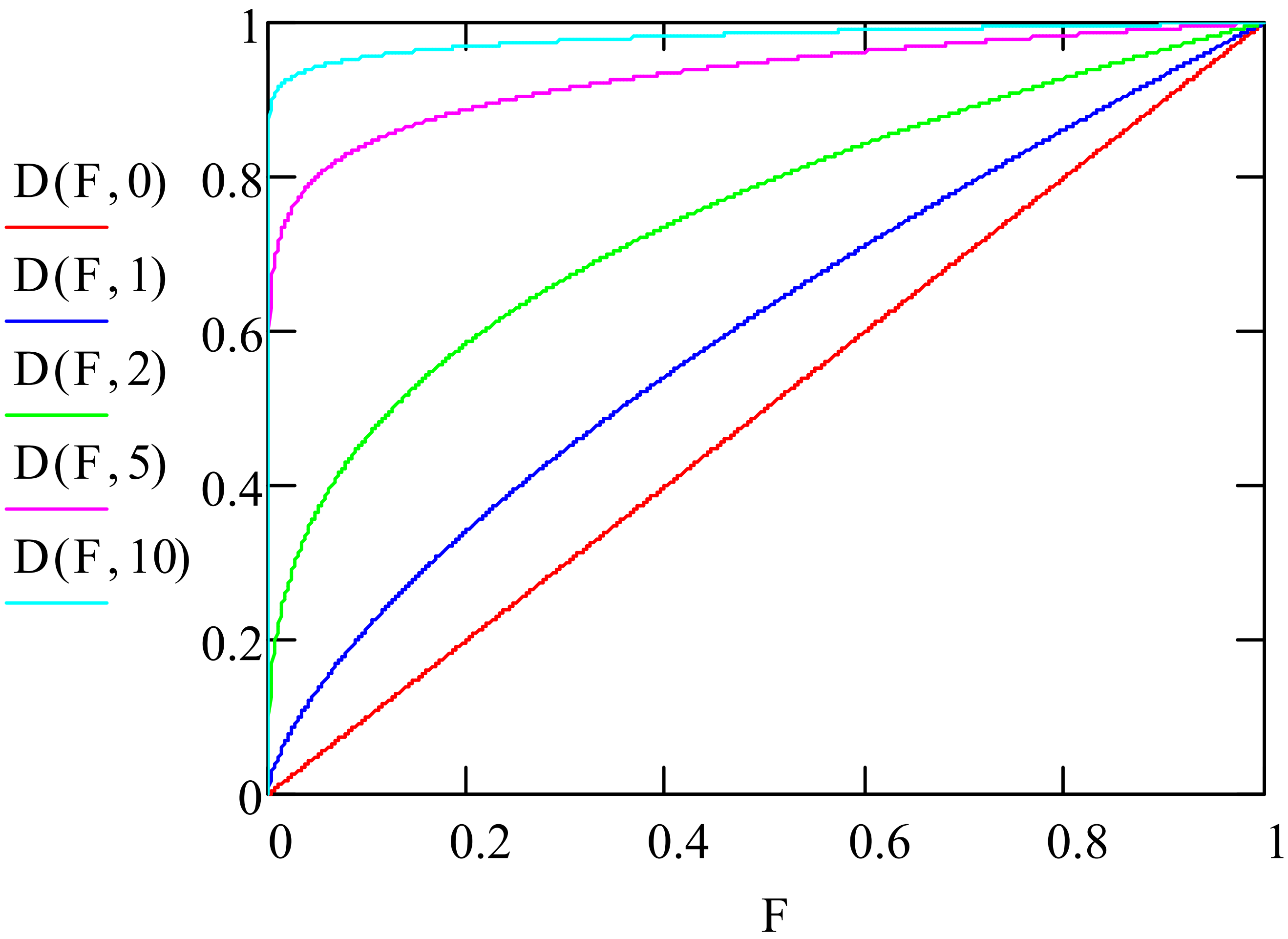


$$D(F,q) := F^{\left(1+\frac{q^2}{2}\right)^{-1}} \qquad q := 0, 0.1 .. 30$$

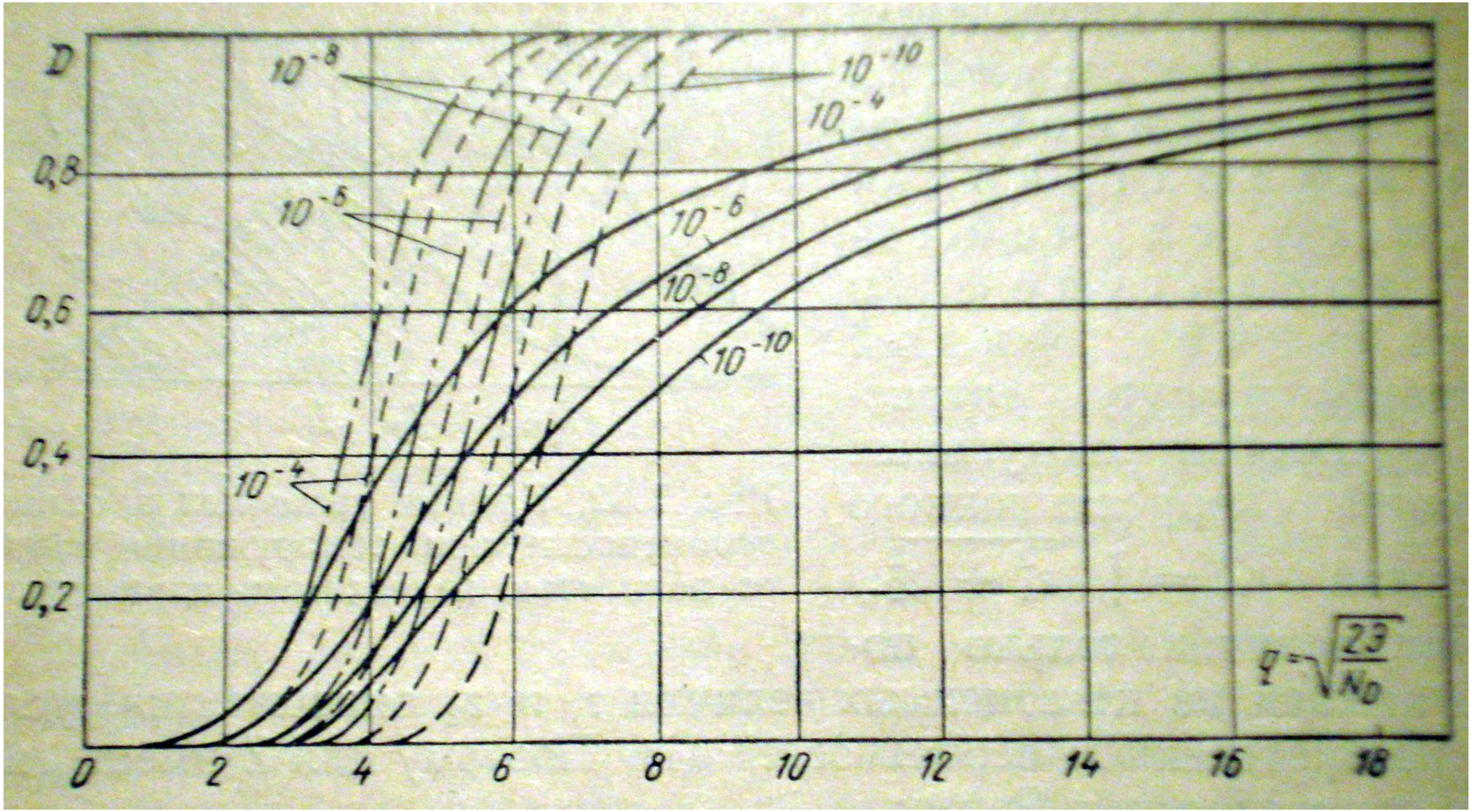


Operating characteristics $D = F^{\left(1+\frac{q^2}{2}\right)^{-1}}$

$F := 0, 0.001 .. 1$



Detection Curves



Detection of a pulse packet (burst of pulses)
Coherent burst with known parameters

- Known burst can be considered as a completely known single signal of a <u>complex</u> shape, which is defined by the shape of a given packet of pulses. Model 5.
- It is a special case of the Model 1.
- All formulas obtained earlier are valid. But:

$$E = \int s^2(t)dt \qquad\qquad E = \sum_{i=1}^{N} E_i$$

E_i is energy of i-th pulse

The structure is the same (correlation receiver) but one should apply a reference copy of signal as the packet of pulses:

$$s(t) = \sum_{k=1}^{N} A(t)\cos[\omega_0 t + \varphi(t) + \varphi_0]$$

$$\varphi_{01} = \varphi_{02} = \ldots = \varphi_{0N}$$

$$t = t_0, \ldots, t_0 + T$$

(integration from *0* to *T*)

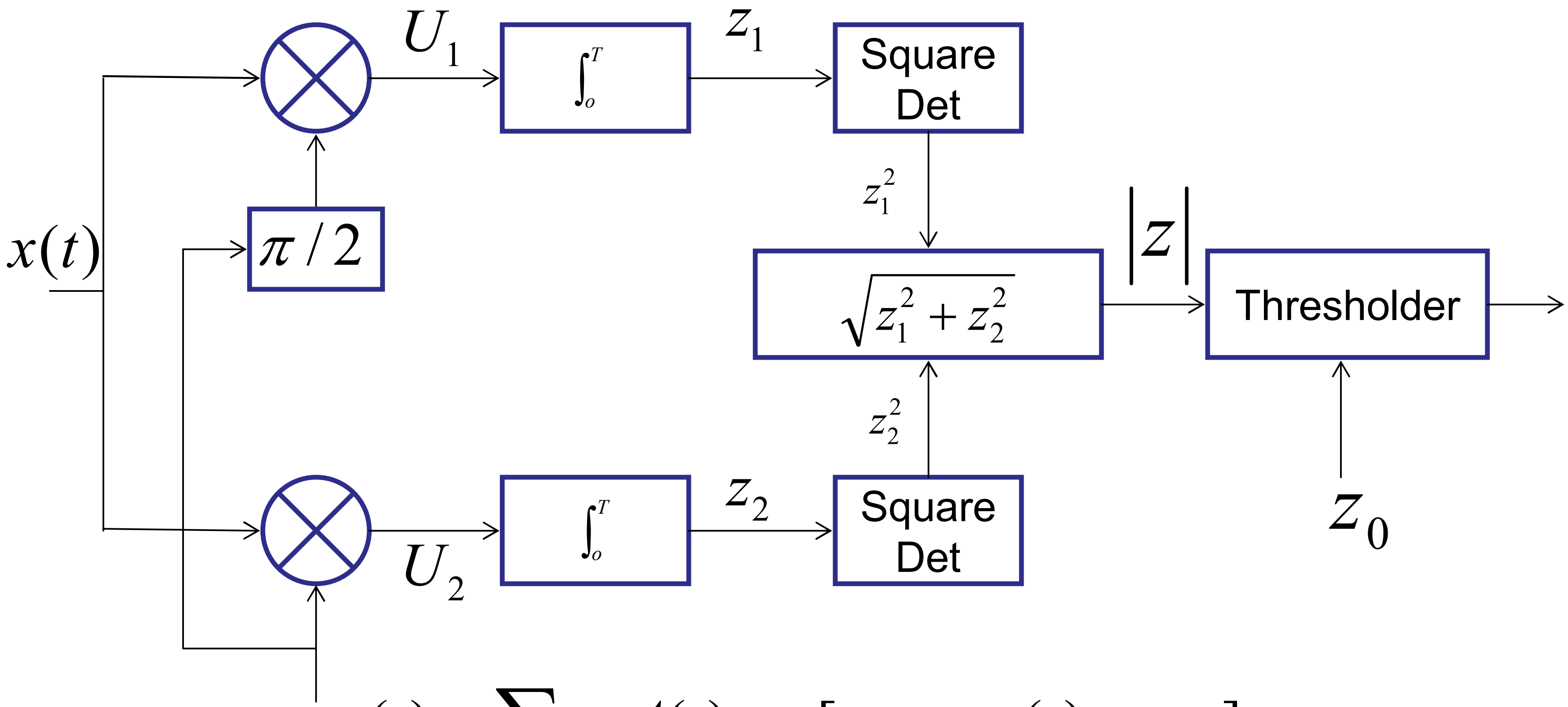
Maximum value of the response will be at time T, in the moment of the end of the packet

All characteristics of detection will be the same as before, if the energy E is the energy of the whole packet of pulses (burst)

Detection of a pulse packet (burst of pulses)

A burst of pulses with amicable fluctuations

- In this case there are no fluctuations in the limits of the packet but impulse signals fluctuate from burst to burst. Model 6.2.
- That means that the amplitude of pulses in the burst is random (unknown), but it is the same for all pulses of given burst.
- Such signal can be considered as a single signal of complex shape with unknown amplitude. If initial phase is also random, we can reduce this case to the case of random amplitude and phase. (Again 2 quadrature channels)



$$s(t) = \sum \alpha_k A(t) \cos[\omega_0 t + \varphi(t) + \varphi_{0k}]$$

In case of amicable fluctuations $\alpha_1, \dots, \alpha_N$ are related between themselves but unknown.

Detection of a pulse packet (burst of pulses)

A burst of pulses with independent fluctuations

- In this case, the model of the signal is almost the same, but $\alpha_1,...,\alpha_N$ are fluctuated randomly and independently in the limits of the burst.
- For the k-th pulse, the likelihood ratio can be found similarly as for the single signal with random amplitude and phase

$$\lambda_k(x) = \frac{N_0}{E_k + N_0}\exp[\frac{z_k^2}{N_0(E_k + N_0)}]$$

- At independent fluctuations of pulse amplitude, the likelihood ratio (LHR) for the whole signal can be represented as a product of LHRs for separate pulses

$$\lambda(x) = \prod_{k=1}^{N} \lambda_k(x)$$

A burst of pulses with independent fluctuations

- As a result of multiplication, there will be a sum of responses on each pulse of the packet in the index (exponent of power). In this case one should calculate

$$Z(T) = \sum_{i=1}^{N} z_i$$

- Similarly to the case of a single signal, the response $Z(T)$ will be proportional to the energy of the input signal, that is, the packet of pulses in particular case.

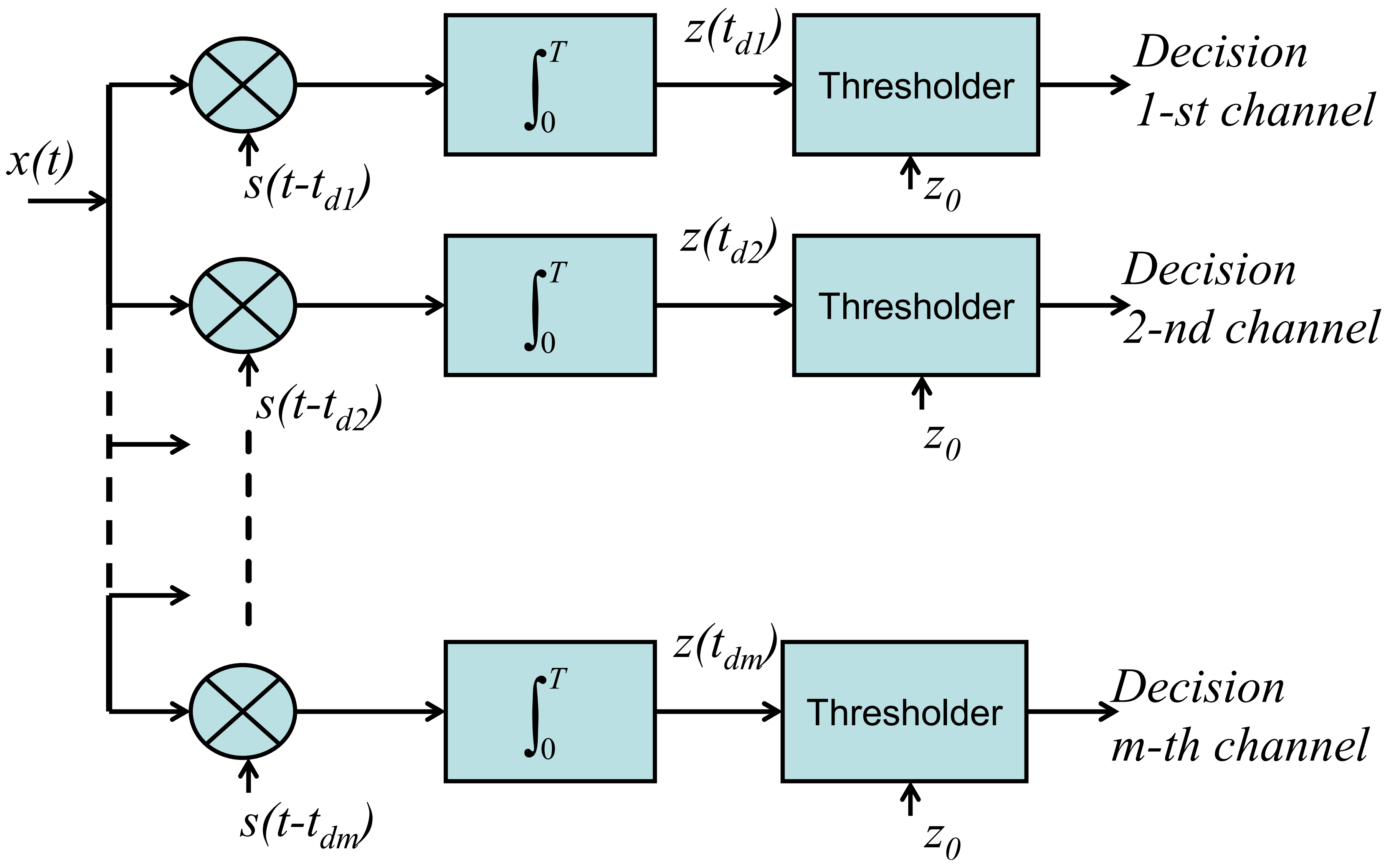
- The result should be compared with the threshold z_0 .

Detection of signals with unknown arrival time

- If the time of signal arrival is unknown, the correlation integral is a function of the unknown time τ

$$z(\tau + T) = \int_0^T x(t)\, s(t-\tau)\, dt$$

- It is necessary to apply a copy of the expected signal to the multiplier synchronously and in-phase with receiving signal. But we do not know the time delay $\tau = t_d$ in advance !!!
- So, we need many copies $s(t)$ that are shifted relative to each other by the interval, defined by the range resolution.


x(t)
s(t-td1)
s(t-td2)
s(t-tdm)
z(td1)
z(td2)
z(tdm)
Thresholder
Thresholder
Thresholder
z0
z0
z0
Decision
1-st channel
Decision
2-nd channel
Decision
m-th channel

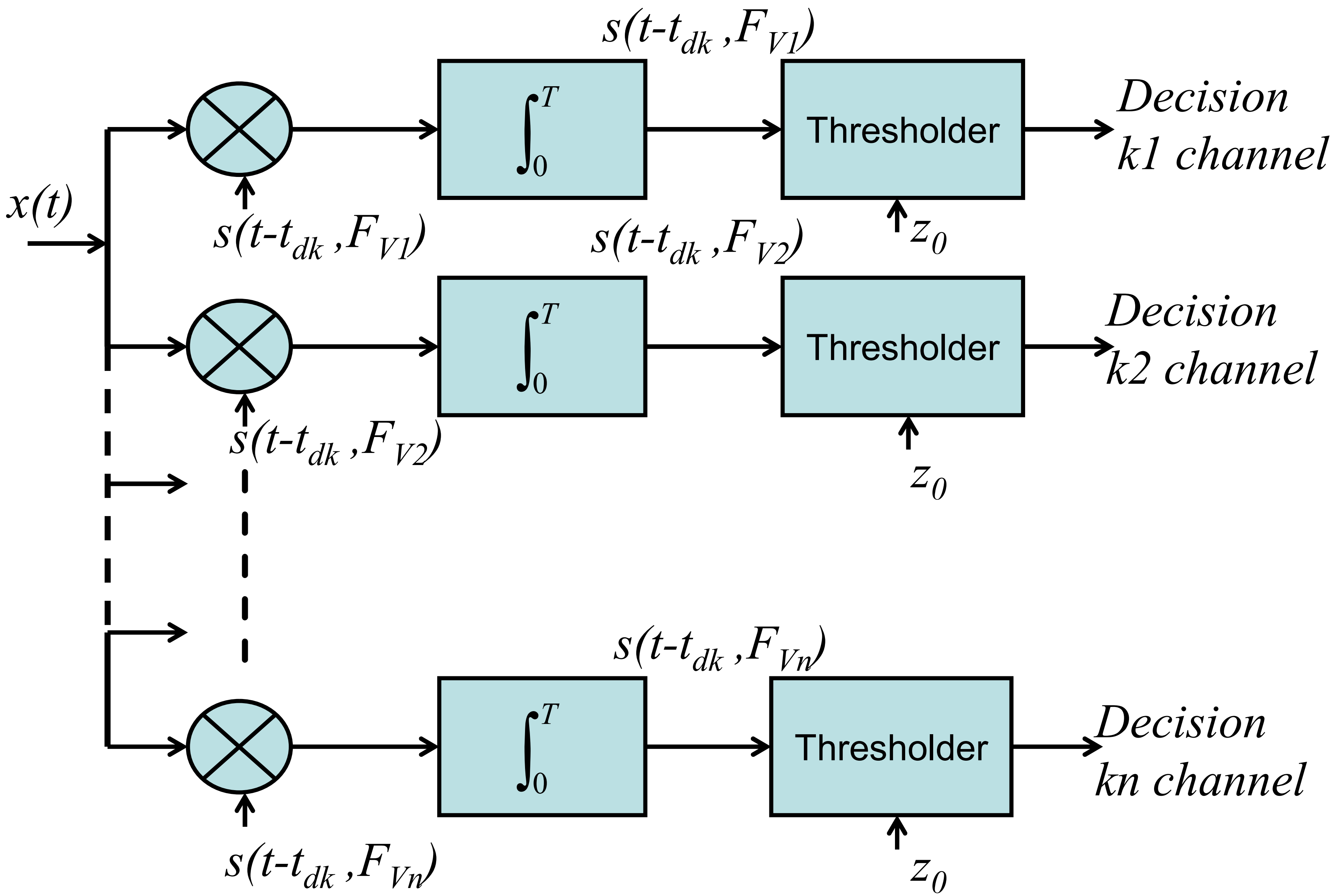
- In case of random signals – correspondingly quadrature structure.
- In fact, we have proceeded to the observation over time in the limits of

$$t_{d\,min} \dots t_{d\,max}$$

- However, in addition to unknown time delay, it can be also an unknown Doppler shift.

Detection of signals with unknown frequency shift

- In this case, we should proceed also to the observation over Doppler frequency in the limits of $F_{v\,min} \ldots F_{v\,max}$
- In other words, we can build multi-channel correlation schemes in time domain and in frequency domain
- For each k-th temporal channel many frequency channels can be created.


x(t)
s(t-t_dk ,F_V1)
s(t-t_dk ,F_V1)
Thresholder
Decision
k1 channel
z_0
s(t-t_dk ,F_V2)
s(t-t_dk ,F_V2)
Thresholder
Decision
k2 channel
z_0
s(t-t_dk ,F_Vn)
s(t-t_dk ,F_Vn)
Thresholder
Decision
kn channel
z_0

- Number of channels: $n*m$; $m=(tmax-tmin)/\Delta t$; $n=(Fmax-Fmin)/\Delta F$; $\Delta t=2\Delta R/C$; $\Delta F=2\Delta V/\lambda$

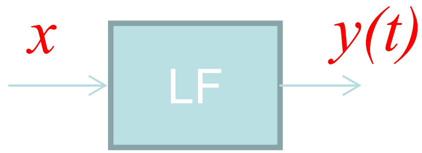
- Correlation schemes of radar detectors require big number of channels in order to detect signal which arrive at different time (to scan all range from R min to R max).
- That is why in many cases it is preferable to use radio engineering devices which are invariant with respect to arrival time.
- In this case we get a possibility to use a single-channel schemes of detectors

Matched filters

- The notion of MF is based on the knowledge of linear radio engineering circuits or electrical circuits and signals.
- Response $y(t)$ of a LF on the impact x at a point of time t is defined by Duhamel integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$h(t)$ is pulse-response characteristic of the circuit (LF).



Demand: the function $y(t)$ should be equal to the CI z to within a constant factor k_0 at the point of time $t=T$, that is, at the time of finishing the useful signal, when

$$y(T) = \int_{-\infty}^{T} x(\tau) h(T-\tau) d\tau$$

We can substitute t instead of τ

$$y(T) = \int_{-\infty}^{T} x(t) \underbrace{h(T-t)}_{k_0 \cdot s(t)} dt$$

Thus, in order to satisfy the equality $\boxed{y(T) = k_0 z}$

$$z = \int_{0}^{T} s(t)\, x(t)\, dt$$

we need $h(T-t) = k_0 \cdot s(t)$

(Lower limit of int. can be 0 because signal s(t) begins in t=0)

Moreover, the condition $h(T-t) = k_0 \cdot s(t)$

is equivalent to

$$\boxed{h(t) = k_0 \cdot s(T-t)}$$

Proof:

$$T - t = \xi; \qquad t = T - \xi;$$

$$\boxed{h(\xi) = k_0 \cdot s(T-\xi)}$$

Thus, the cross-correlation function at the time of the end of the useful signal is produced at the output of the linear filter with the impulse response, which is a mirror image of the desired signal (up to a constant factor).

Linear filters are described by impulse response and frequency response

$$y(t) = \int_{-\infty}^{T} x(t)h(T-t)dt = \int_{-\infty}^{\infty} x(t)h(T-t)dt$$

Formula of filtering in the time domain, or a convolution integral

$$y(t) = \int_{-\infty}^{\infty} S_x(f)K(f)e^{j2\pi ft}df$$

Formula of filtering in the frequency domain

Impulse response and frequency response are related by Fourier transforms

$$h(t) = \int_{-\infty}^{\infty} K(f) e^{j2\pi ft} df$$

$$K(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt$$

Definition of Matched Filter in Radar

- Matched filter with respect to the expected signal is a filter that takes into account the shape of the signal and is capable to delivering on its output consistently over time the values proportional to the correlation integral at different time delays of the signal.
- Math: $$y(T+t) = C \cdot z(t)$$

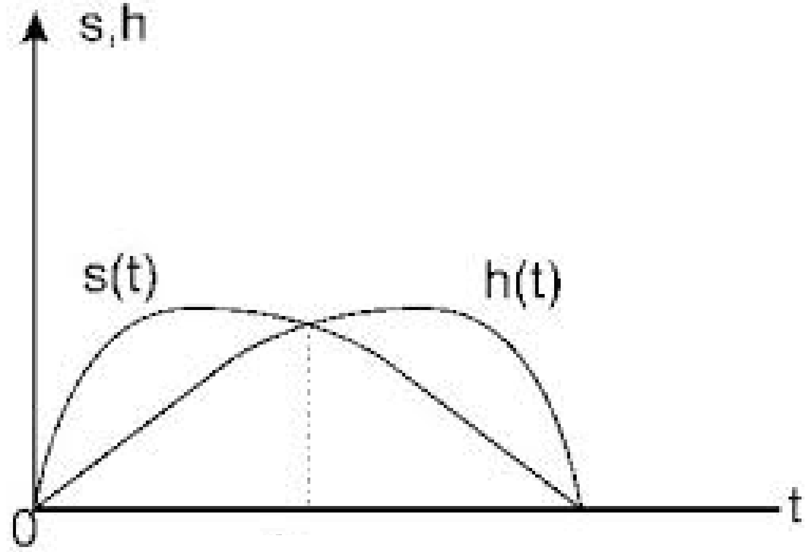
with C – factor of proportionality; T – delay in the filter itself.

- The farther the target, the greater the delay of the reflected signal, so, the response of the filter to the signal is later.
- The delay T in the filter itself is necessary to account all information that arrives during signal duration.

Impulse response of a MF

$$h_{MF}(t) = k_0 \cdot s(T - t)$$

Impulse response of a MF is constructed by mirroring the expected signal



T/2

Frequency response of a MF

can be found from impulse response:

$$K_{MF}(f) = k_0 \int s(T-t) e^{-j2\pi ft} dt$$

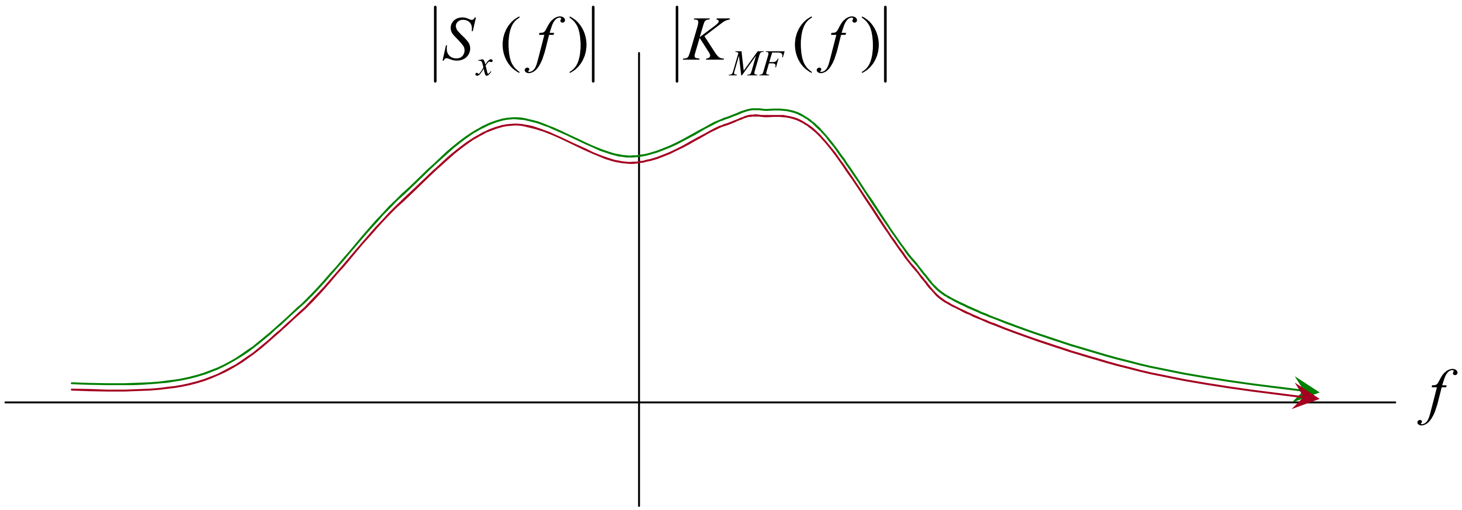
Change of variable $t=T-\tau$ leads to:

$$K_{MF}(f) = k_0 S_x^*(f) e^{-j2\pi fT}$$

Frequency response of the matched filter is proportional to the product of the complex conjugate value of the spectral density of the expected signal and the factor of the signal delay in the filter

Amplitude-frequency characteristic of MF corresponds to the amplitude-frequency spectrum of the signal

$$\left|K_{MF}(f)\right| = k_0\left|S_x(f)\right|$$



Phase-frequency characteristic of MF

corresponds to the amplitude-frequency spectrum of the signal

$$\arg K_{MF}(f) = -\arg S_x(f) - 2\pi fT$$

- Phase response consists of two components:

- the argument of the function $S_x^*(f)$, which is reverse in respect to the phase spectrum of the signal;

- a phase factor $e^{-j2\pi fT}$.

- The first one provides a summation of all frequency components 'in phase', in the point of time T, when the signal is ended.
- The second factor corresponds to the delay T in the filter. So, at time T there is a maximum value of the response, which is numerically equal to the signal energy.

- At the output of MF, the peak voltage of the signal DOES NOT DEPEND on the shape and bandwidth of the signal:

$$y_{s\ peak} = k_0 \int_{-\infty}^{\infty} \left|S_x(f)\right|^2 df = k_0 \int_{-\infty}^{\infty} S^2(t)dt = k_0 E$$

E – signal energy on the unity resistor

- Mean square of noise voltage (mean power)

$$y_n^2 = k_0 \int_{-\infty}^{\infty} N(f)\left|K_{MF}(f)\right|^2 df$$

$N(f) = \boxed{N_0/2}$, *if* $-\infty < f < \infty$. $\left|K_{MF}(f)\right| = k_0 \left|S_x(f)\right|$, so:

$$\int_{-\infty}^{\infty} \left|K_{MF}(f)\right|^2 df = k_0^2 \int_{-\infty}^{\infty} \left|K_{MF}(f)\right|^2 df = \boxed{k_0^2 E}$$

Thus: $$y_n^2 = k_0^2 E \frac{N_0}{2}$$ Hence peak SNR is: $\dfrac{y_{s\ peak}}{y_n}$

Peak SNR

$$\frac{y_{s\ peak}}{y_n} = \frac{k_0 E}{k_0 \sqrt{\frac{EN_0}{2}}} = \sqrt{\frac{2E}{N_0}} = q$$

It coincide with parameter of detection !

$$m_{dsc} = q = SNR(1)$$

– coefficient of discrimination (required power SNR per single pulse); if losses are absent,

But losses in real circuits and devices should be taken into account.

Coefficient of discrimination – the required SNR

$$m_{dsc} = q \cdot \prod_{i=1}^{n} \alpha_i$$ - in case of detecting a single signal

$$m_{dsc} = \frac{q}{N} \cdot \prod_{i=1}^{n} \alpha_i$$ - in case of detecting the coherent packet of N pulses

q is defined on given D and F;

α_i takes into account different kinds ($i=1,\ldots,n$) of losses.

Losses totally: $\prod_{i=1}^{n} \alpha_i$ or in dB: $\sum_{i=1}^{n} \alpha_{i\ dB}$

$$m_{dsc} = SNR(1)$$

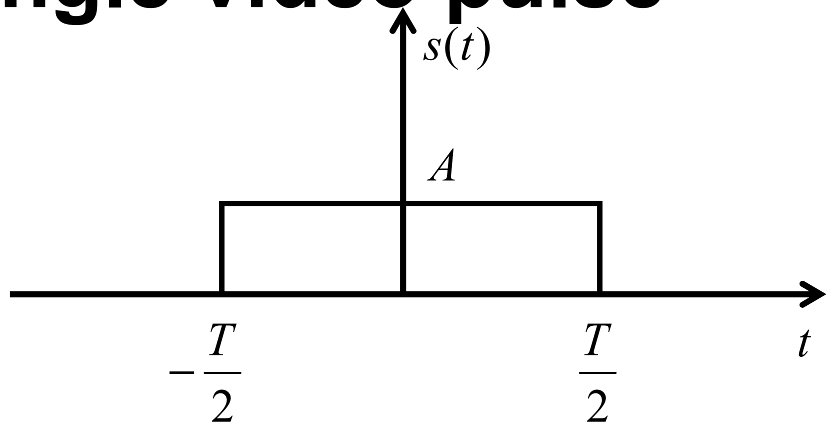
Sources of losses α_i

- Loss in detector
- Detuning
- Attenuation in Tx and Rx lines ($\approx$2 dB)
- Loss in antenna (antenna pattern) ($\approx$1.8 dB)
- Loss due to fluctuating target RCS
- Losses due to mismatch (quasi-optimal filter)
- CFAR loss (if any)
- MTI loss (if any)
- Losses related with operator (if any)
- Miscellaneous additional losses

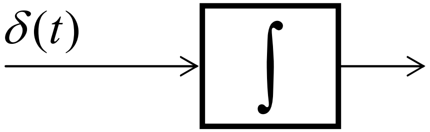
Matched filter for a single video pulse

Signal

$$s(t)=\begin{cases}A, & |t|\le \dfrac{T}{2}\\ 0, & |t|>\dfrac{T}{2}\end{cases}$$



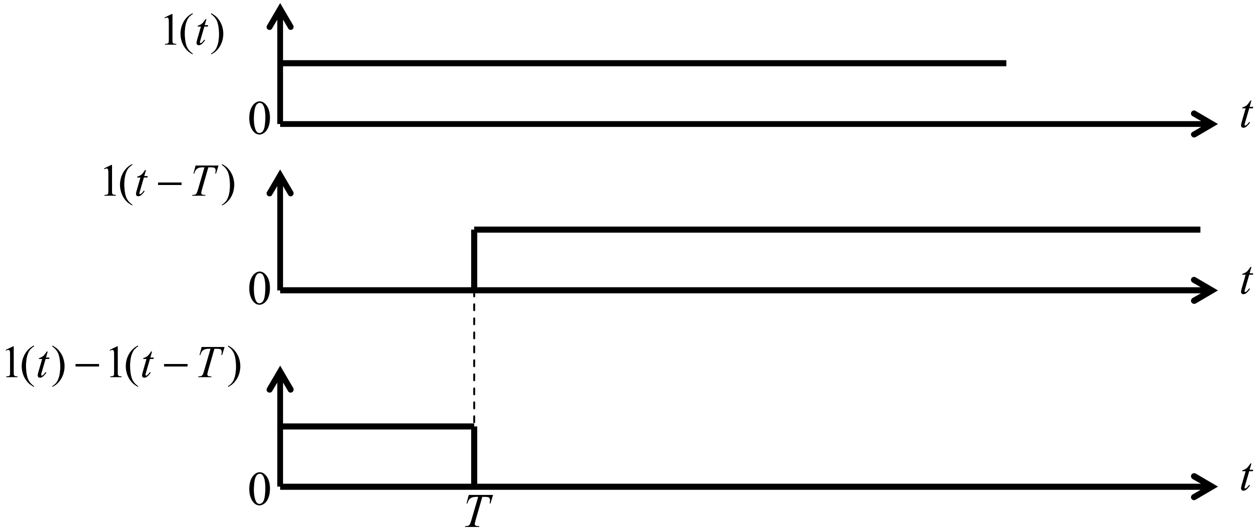
Action of delta-function on an integrator

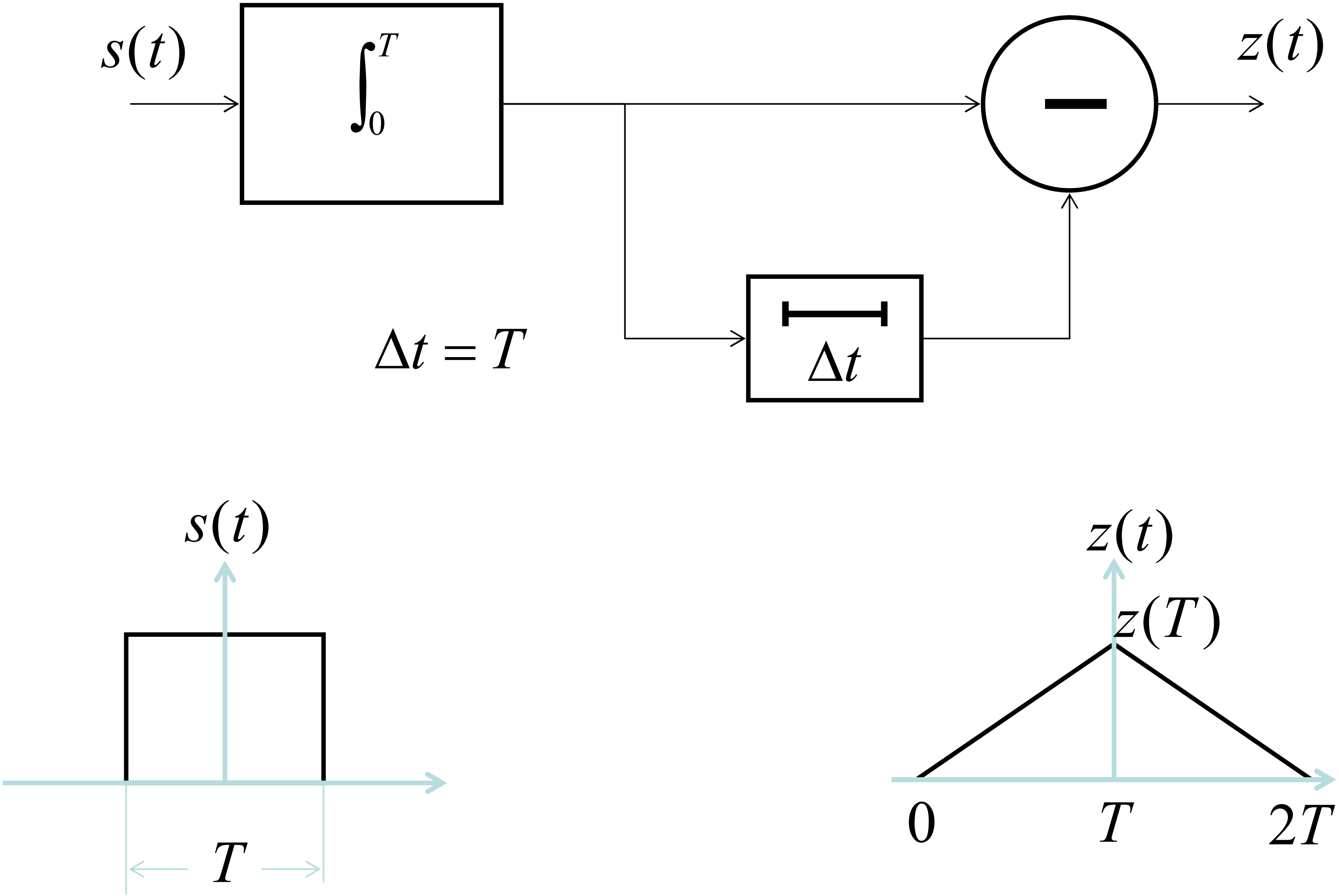


$$\begin{cases}0, & at\ t\le 0\\ 1(t), & at\ t>0\end{cases}$$

unit step of voltage

A rectangular pulse is a difference of unit steps



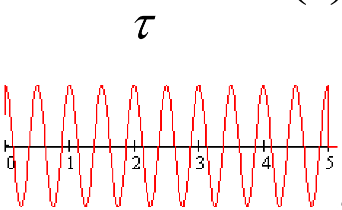


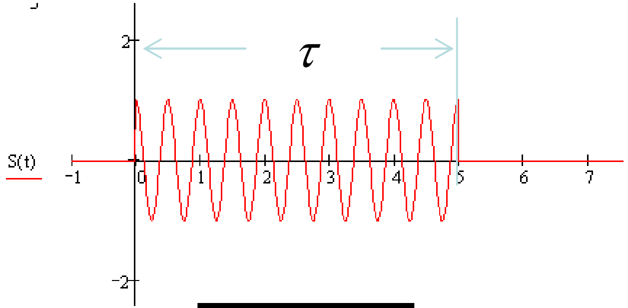
$s(t)$
$\int_0^T$
$z(t)$
$\Delta t = T$
Δt
$s(t)$
T
$z(t)$
$z(T)$
0
T
$2T$

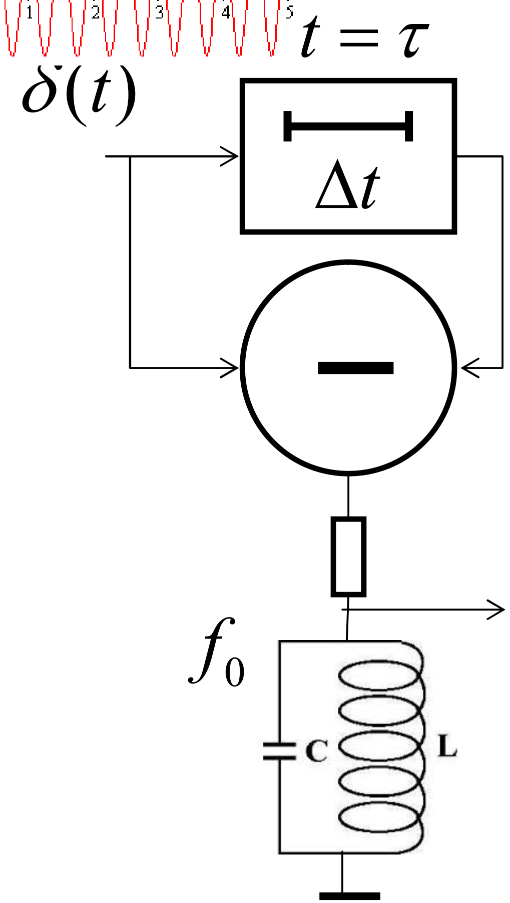
Matched filter for a single RF pulse

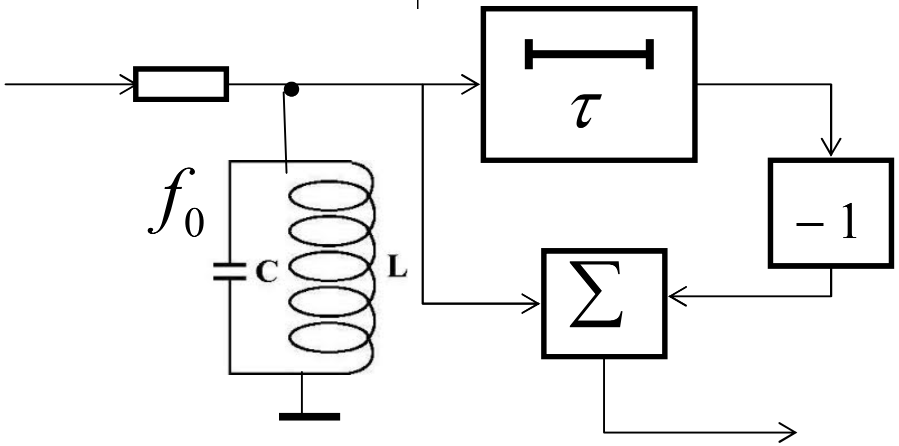
Signal

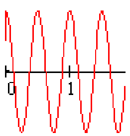
$$s(t)=\begin{cases}\cos 2\pi f_0 t, & |t|\le \dfrac{\tau}{2}\\ 0, & |t|>\dfrac{\tau}{2}\end{cases}$$

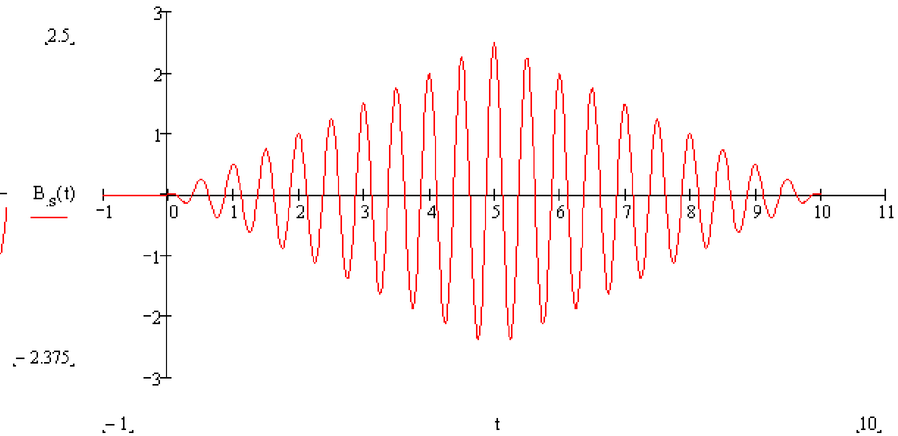












Quasi-optimal filtering

- In practice a quasi-optimal filters are often used.
- It can be done by optimization of filter bandwidth
- In case of band-pass filter (rectangular)

$\Delta f \cdot \tau = 1.37$, and $\text{SNR}_{max} = 0.83q$

that is, only 17 % will be lost ($\approx$1.2 times)

Δf = B – Bandwidth of the receiver with quasi-optimal filter

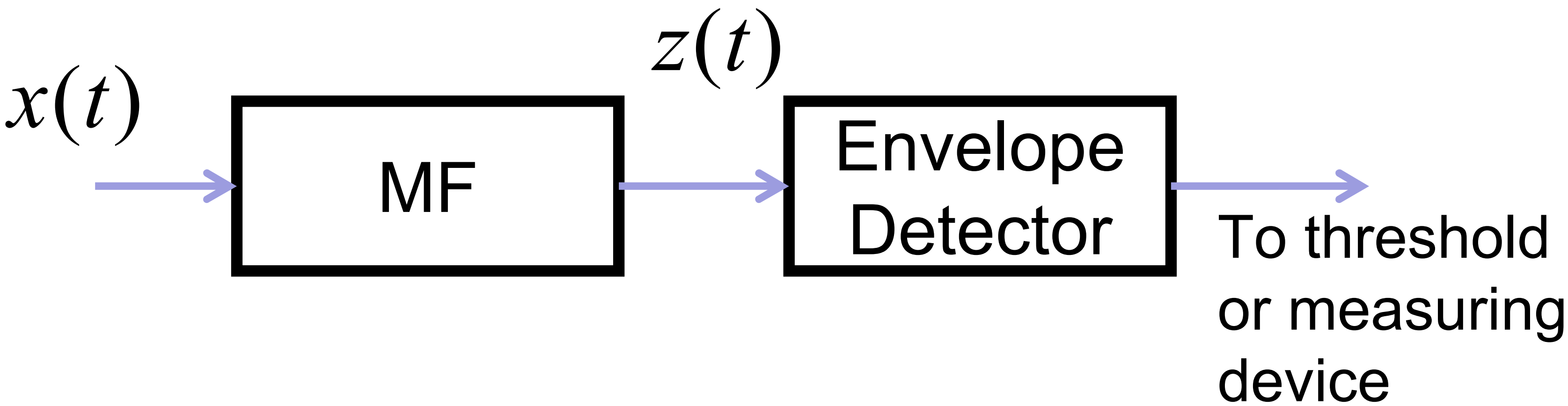
Matched and Non Matched (Quasi-optimal) Filters

Input Pulse Shape	Filter Shape	Optimum $\Delta f \cdot \tau$	Loss in SNR (dB)
Rectangular	Rectangular	1.37	0.85
Rectangular	Gaussian	0.72	0.49
Gaussian	Rectangular	0.72	0.39
Gaussian	Gaussian	0.44	0
Rectangular	Single tuned circuit	0.4	0.88
Rectangular	2 cascaded tuned ccts	0.613	0.56
Rectangular	5 cascaded tuned ccts	0.672	0.5

Let’s go back to Matched Filter

- It can be shown that the amplitude of the signal at the output of MF determined the module value of the correlation integral.
- It is necessary at optimal detection of a signal with random initial phase (amplitude and phase).
- That is why, a MF for a signal with arbitrary amplitude and initial phase can be used for detection signals with any initial phases and amplitudes.

- In order to proceed from instantaneous values of voltage to the amplitude value, the structure of MF detector includes the envelope detector



Pulse Trains

- The relationships developed earlier between $q =$ SNR, D and F apply to a single pulse only.
- As a search radar scans past a target, it will remain in the beam sufficiently long for more than one pulse to hit the target. The number can be calculated using the following formula:

$$n_b = \frac{\theta f_r}{\Omega_A} = \frac{\theta f_r}{6\varpi_m}$$

where n_b – Hits per scan

θ – *Azimuth beamwidth (deg)*

Ω_A – Azimuth scan rate (deg/s)

ω_m – Azimuth scan rate (rpm)

- For a long-range ground based radar with an azimuth beamwidth 1.5°, a scan rate of 5rpm, and a pulse repetition frequency of 30Hz the number of pulses returned from a single point target is 15.
- The process of summing all these hits is called integration, and it can be achieved in many ways.

MF for packets of pulses

- Coherent burst
  - MF for coherent burst of equal rectangular pulses
  - What will happen, if MF is substituted by quasi-optimal one?
  - What we should do, if the envelope of the packet is not rectangular?

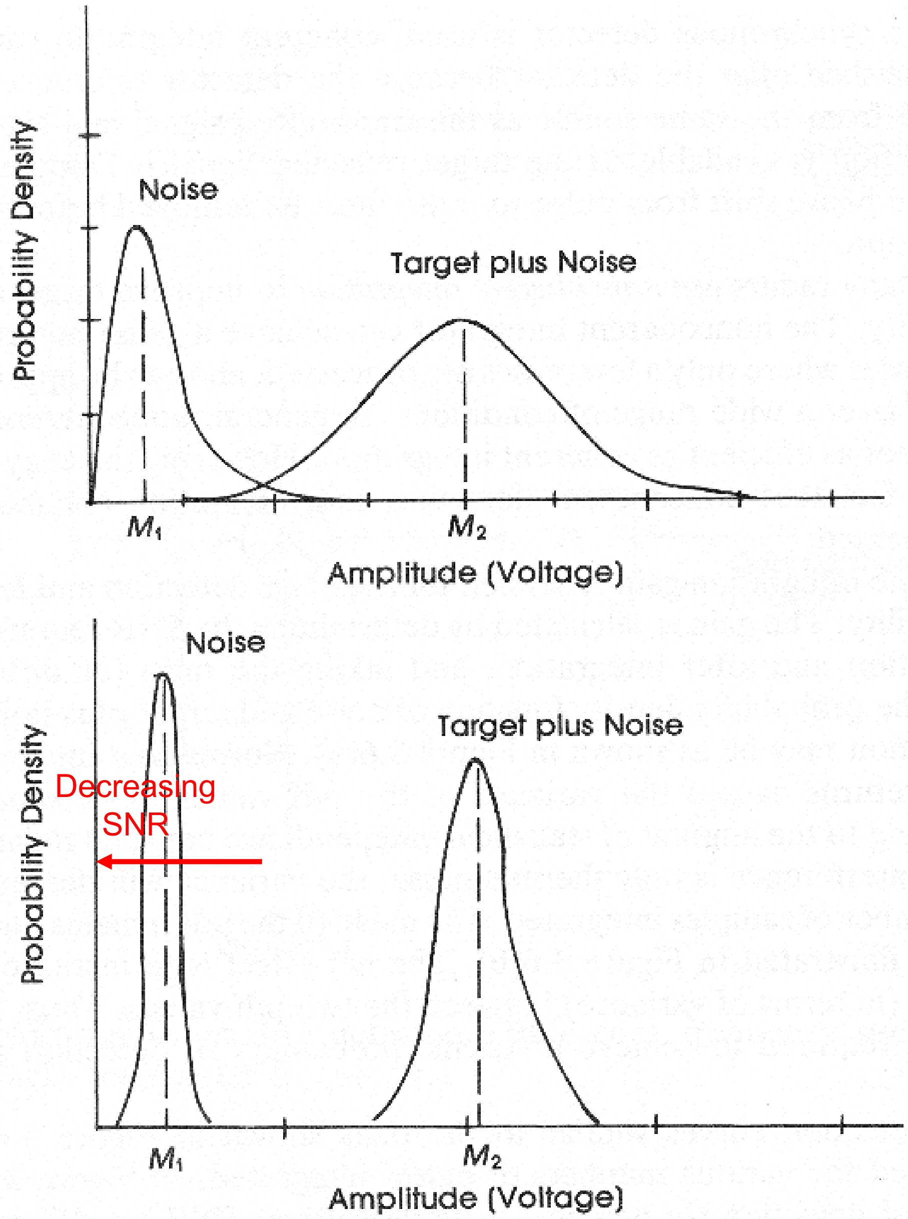
MF for packets of pulses

- Non-coherent burst
  - Weighted postdetection integration
  - Week non-fluctuated packet (square detector) $K_i = A_i^2$
  - Non-fluctuated packet of big amplitude $K_i = A_i$
  - Fluctuating packet

$$K_i = \frac{A_i^2}{A_i^2 + \frac{2}{q^2}} \qquad q = \sqrt{\frac{2E}{N_0}}$$

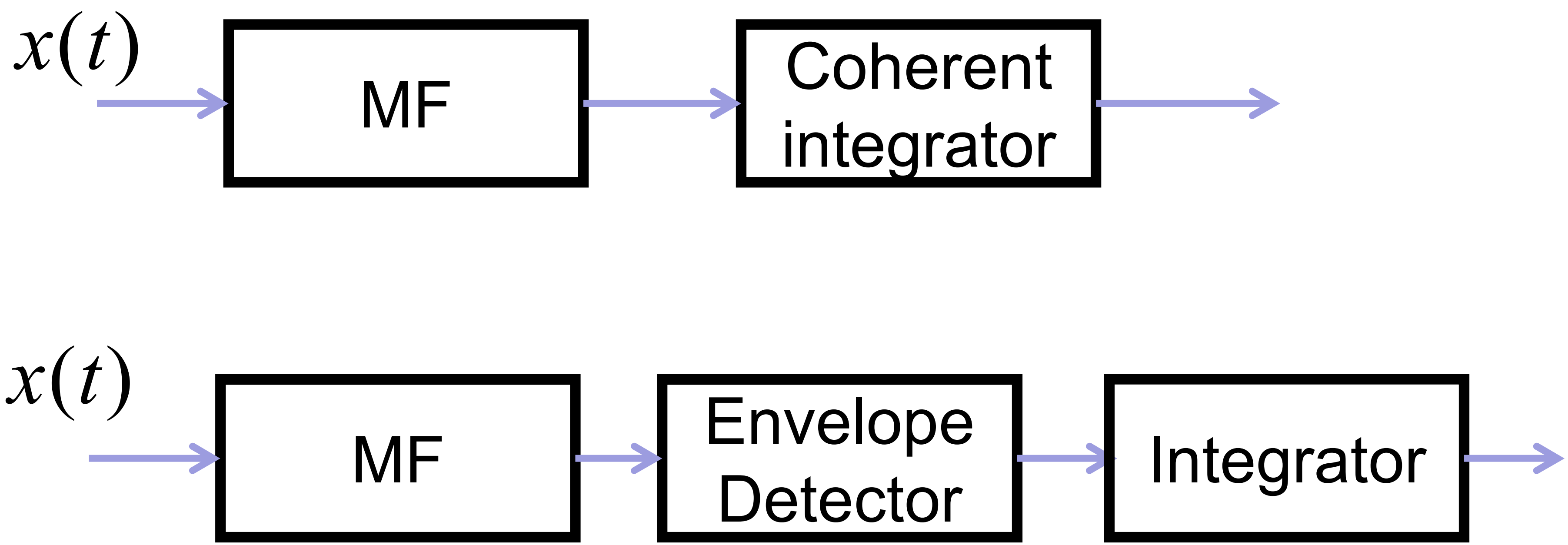
Effect of Integration on PDF's

- Note that though the mean values of both the Noise and Signal+Noise remain unchanged, the variance decreases
- This results in a reduction of the required single pulse SNR, or m_{disc} to achieve a particular D and F

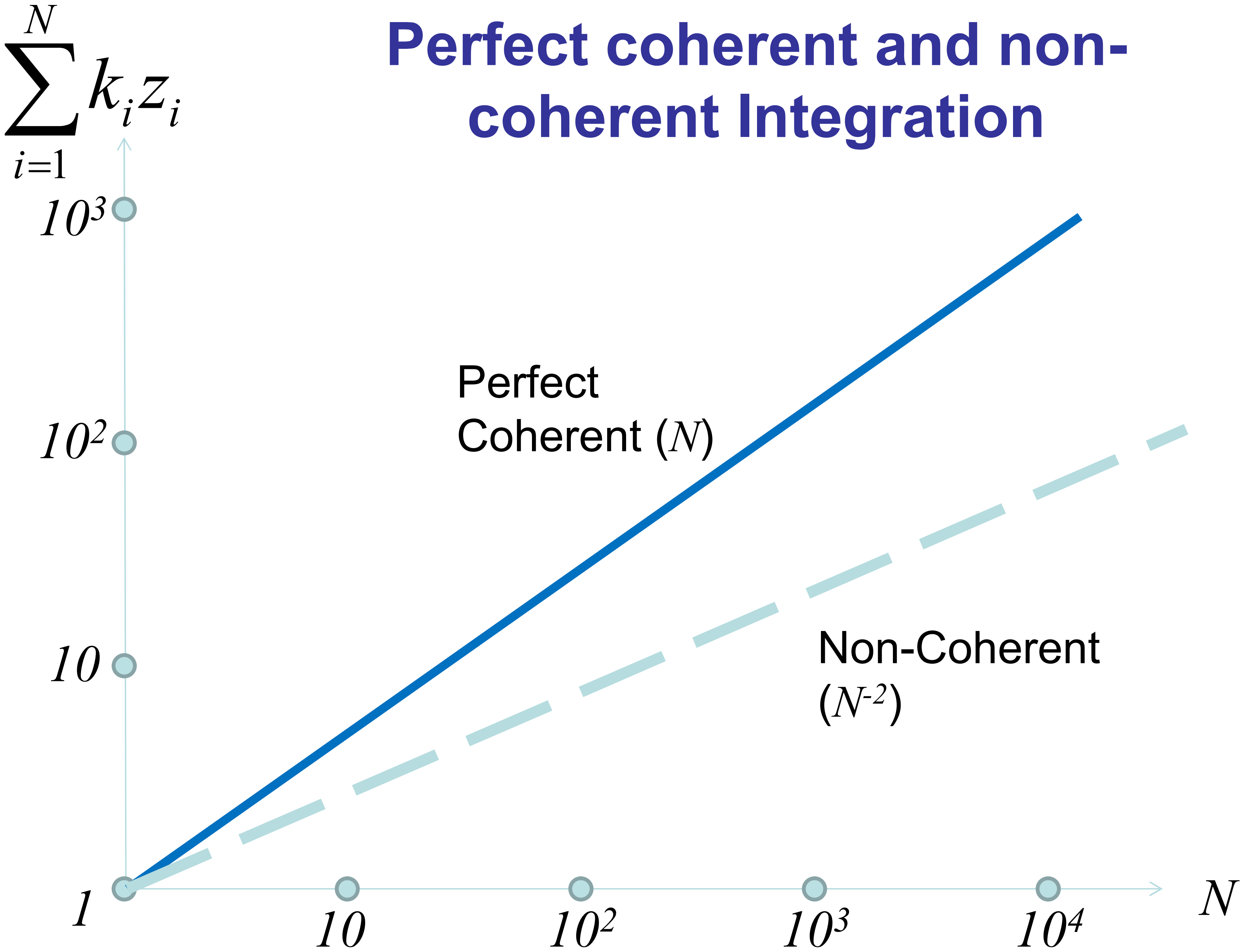


Probability density functions of noise and target-plus-noise (a) before integration and (b) after integration.

Comparison of coherent and non-coherent integration



Perfect coherent and non-coherent Integration



Real Integration Efficiency

- With integration, the required SNR decreases as a function of the number of samples integrated
- However as the single pulse SNR decreases, detector losses increase which result in reduced integration efficiency

$$E_i(n) = \frac{SNR(1)}{nSNR(n)}$$

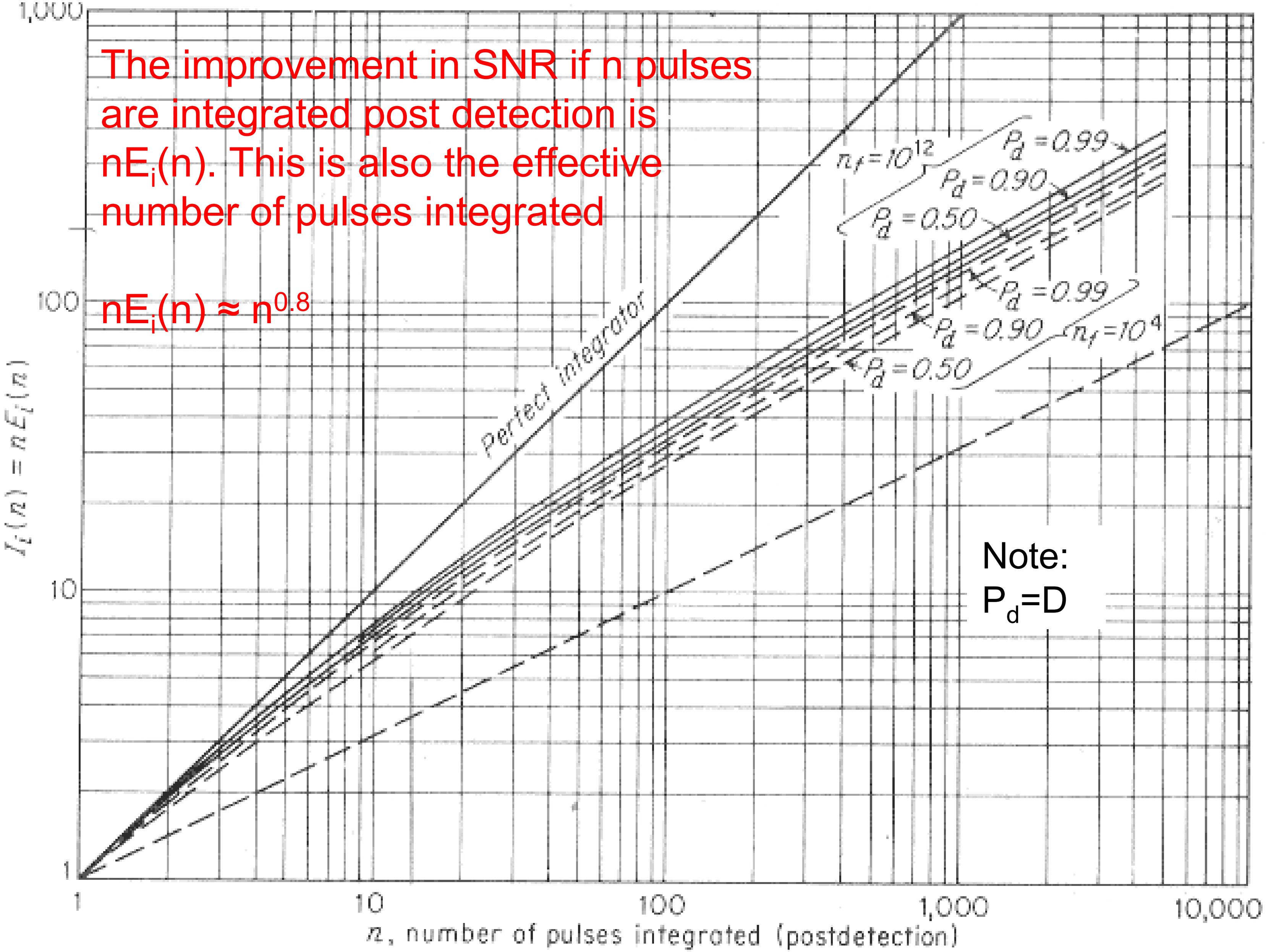
In case of incoherent integration

$$E_i(n) \approx \frac{1}{\sqrt{n}}$$

where: $E_i(n)$ – Integration efficiency

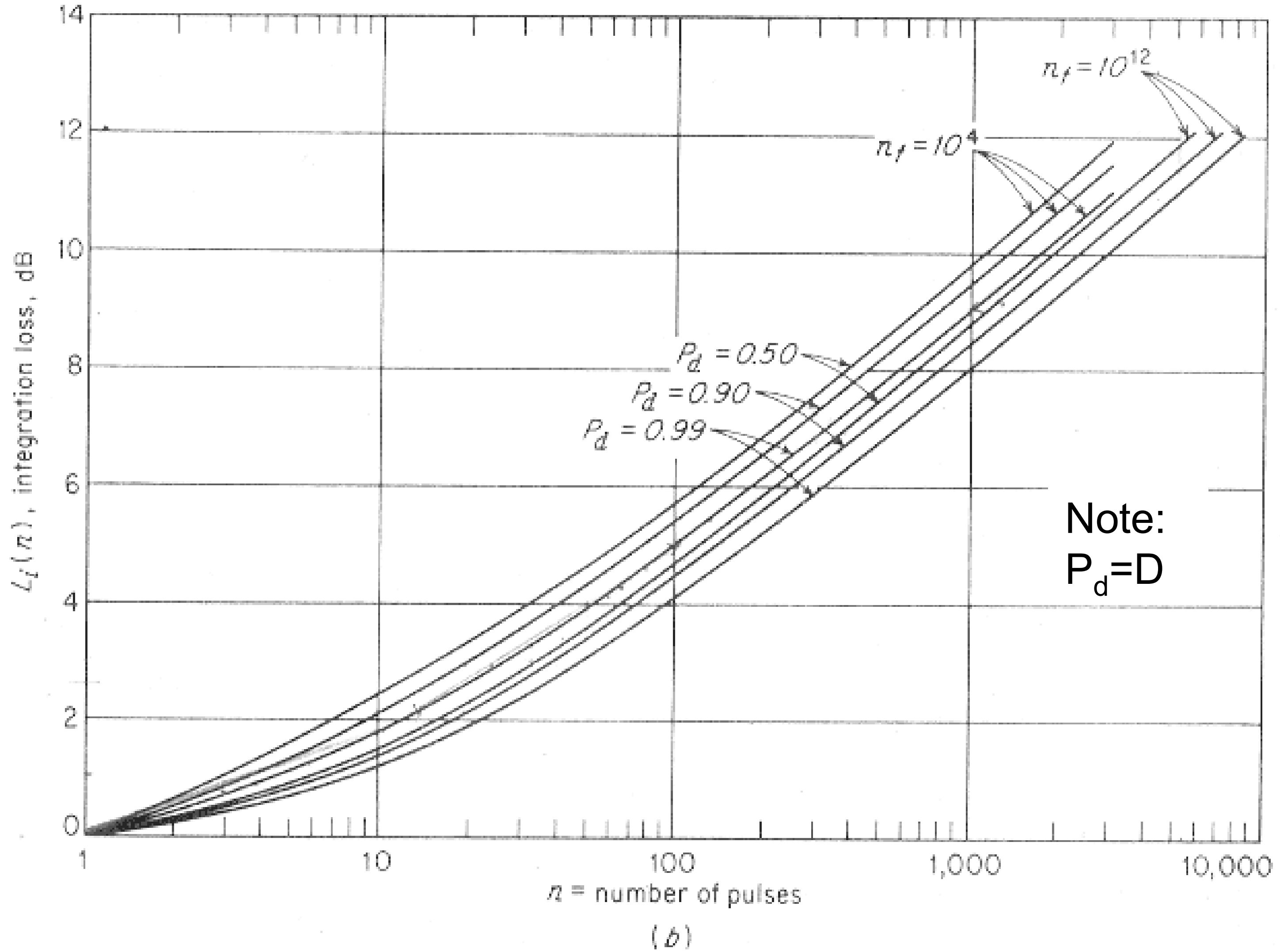
SNR(1) – Single pulse SNR required to produce a specific P_d if there is no integration.

SNR(n) – Single pulse SNR required to produce a specific P_d if n pulses are integrated perfectly.



Integration-improvement factor, square law detector, P_d = probability of detection, $n_f = 1/T_{fa}B$ = false alarm number, T_{fa} = average time between false alarms, B = bandwidth

Integration Loss

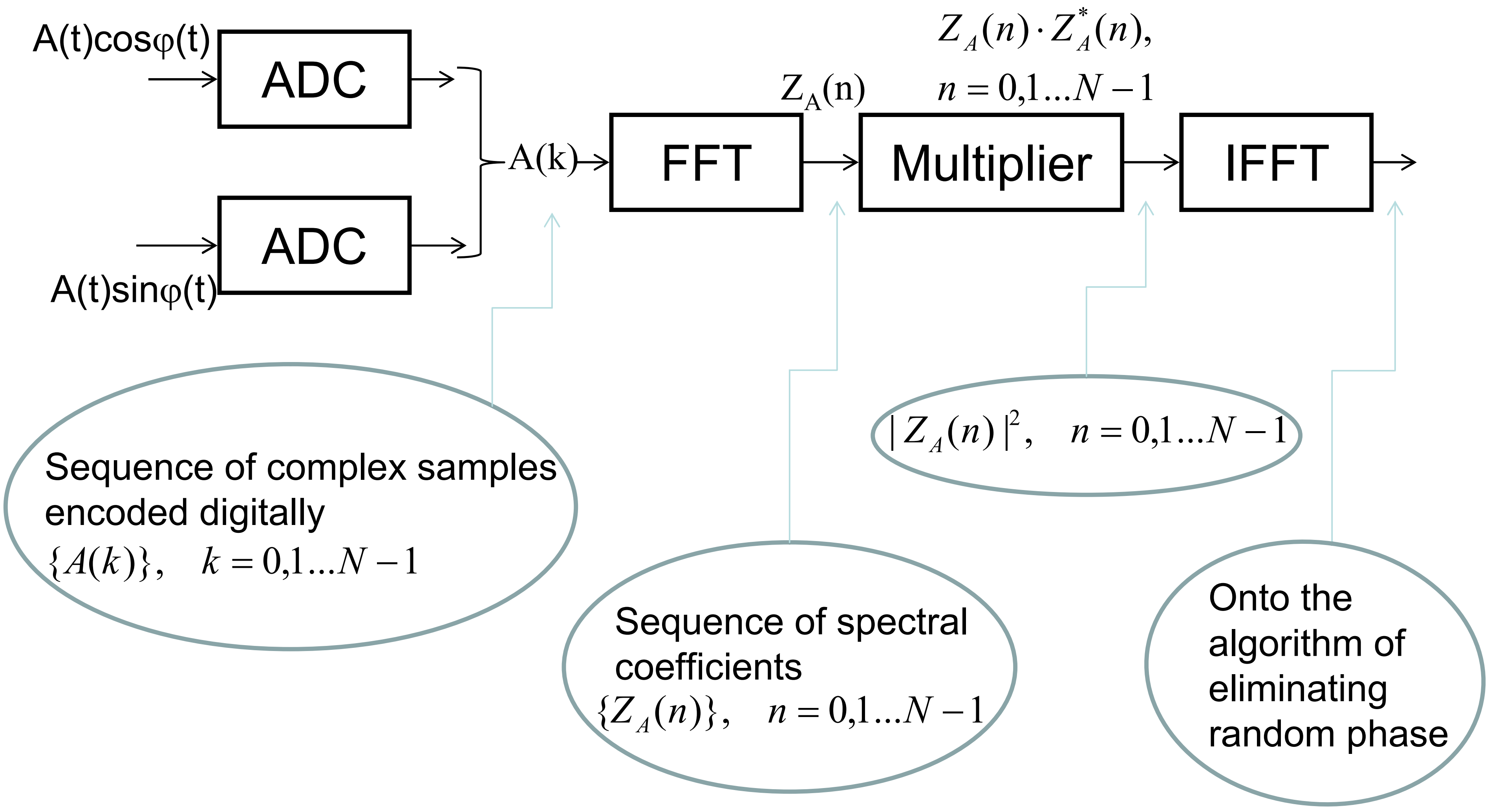


Note:
P_d=D

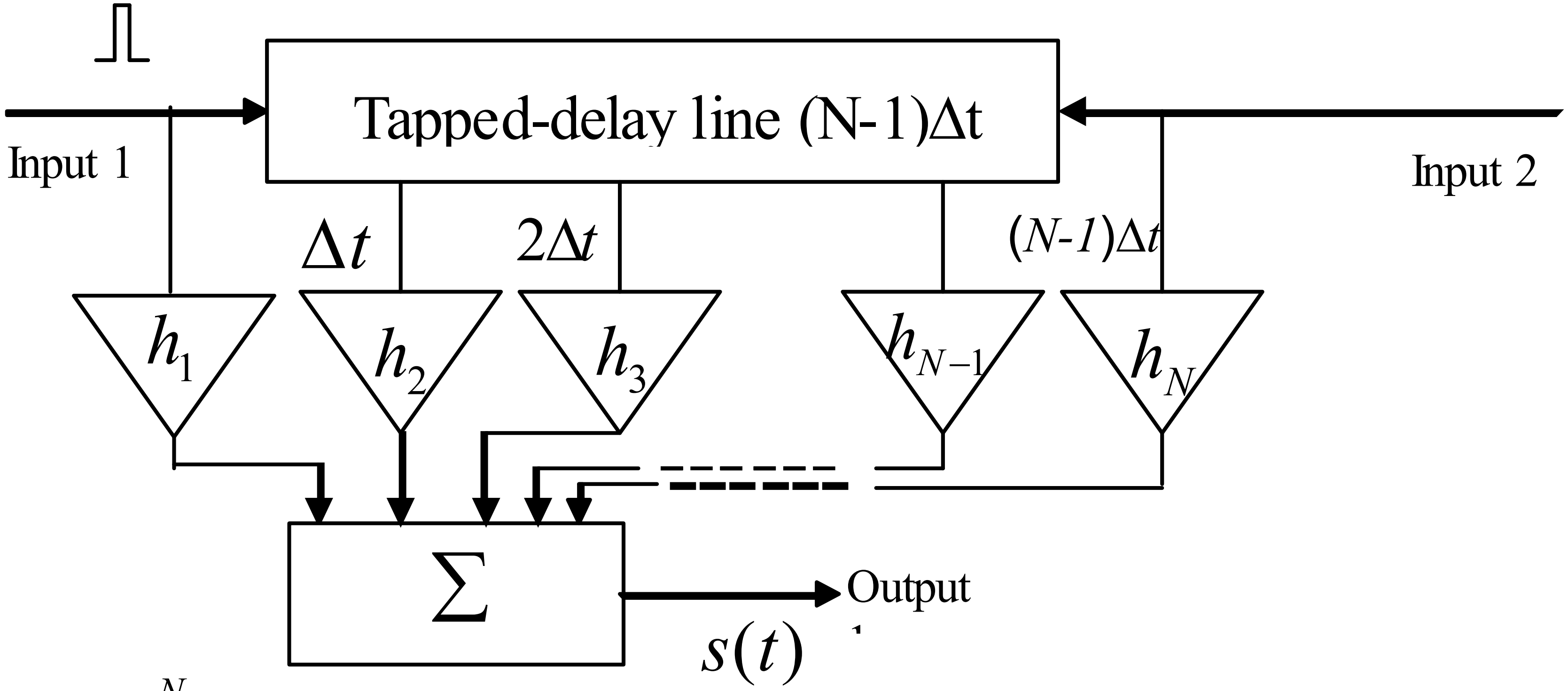
Integration Loss as a function of n, number of pulses integrated

Digital Processing

- Considered general theory is completely suitable for both analog and digital signal processing.
- However the Filter itself normally is synthesized on the basis of spectral approach using FFT.
- MF should deliver at the output, a signal correspondent to correlation integral. So, the algorithm can be built in accordance with the following diagram.


A(t)cosφ(t)
ADC
ADC
A(t)sinφ(t)
A(k)
FFT
$Z_A(n)$
Multiplier
$Z_A(n) \cdot Z_A^*(n),$
$n = 0,1 \ldots N-1$
IFFT
Sequence of complex samples encoded digitally
$\{A(k)\}, \quad k = 0,1 \ldots N-1$
Sequence of spectral coefficients
$\{Z_A(n)\}, \quad n = 0,1 \ldots N-1$
$|Z_A(n)|^2, \quad n = 0,1 \ldots N-1$
Onto the algorithm of eliminating random phase

Universal filter with tapped-delay line



$$s(t) = \sum_{i=1}^{N} h_i s_1(t - i\Delta t)$$

$\Delta t \to \infty$ $s_1 \to \delta(t)$ $s(t) \approx h(t)$

Transversal filter

Нерекурсивний фільтр, Трансверсальний фільтр, Гребінчастий фільтр

Compression of wideband signals

- We could see that MF distorts the shape of the signals but maximizes SNR.
- In case of WB signals the distortion leads to USEFUL EFFECT of COMPRESSION.
- It will be considered in separate topic.

Features of requirements to F and D

1. First feature is related with Great number of resolution volumes
2. Second feature is related with Cyclicity (recurrence) of surveillance

Influence of number of resolution volumes

- If there is ***m*** RVs, then aggregate conditional probability of Correct Undetection in ***m*** volumes is

$$\bar{F}_m = (\bar{F})^m = (1 - F)^m$$

$$F_m = 1 - \bar{F}_m = 1 - (1 - F)^m$$

- Expand into a Taylor series

$$(1 - F)^m = 1 - mF + \frac{1}{2} m(m-1)F^2 - \frac{1}{6} m(m-1)(m-2)F^3 + \ldots$$

If $mF << 1$ $\quad F_m \approx mF$ $\qquad F \approx \frac{F_m}{m} \approx 10^{-4} \ldots 10^{-10}$

Influence of cyclicity (recurrence) of surveillance

- Suppose that optimal signal processing within the cycle of observation is supplemented by the intercycle processing according to logics **“1 of *k*”**.

$$F_k = 1 - (1 - F_1)^k \approx kF_1$$

$$D_k = 1 - (1 - D_1)^k$$

D grows quickly if at increasing *k*!

False Alarm Rate

Another important concept in detection is FAR,

False Alarm Rate. FAR is the number of times a false alarm is expected to occur per second.

It depends on the **F** and the number of tests per second. Stated mathematically:

$$\text{FAR} = \text{F} \times (\text{number of tests/second})$$

Note: a test could be a scan, a dwell, etc.

False Alarm Rate Stabilization

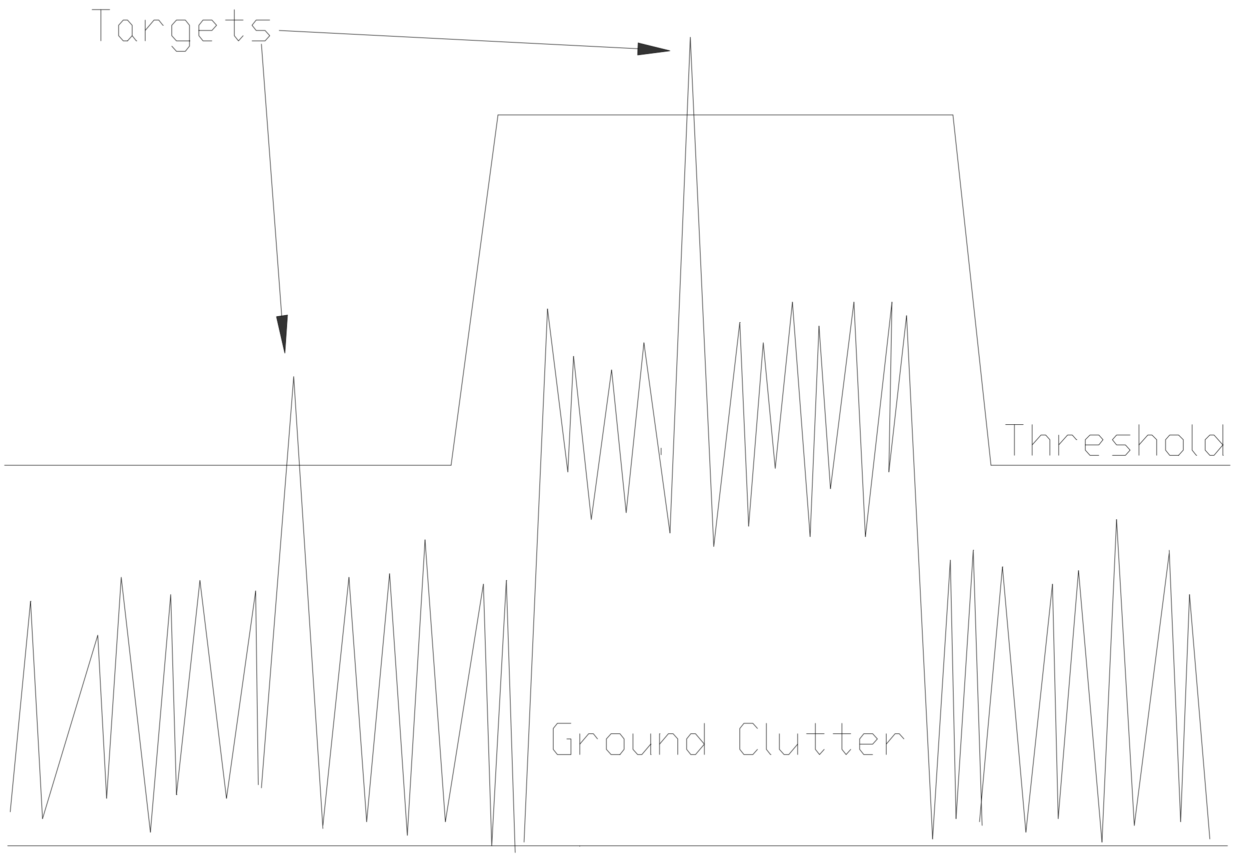
It is desirable that the system automatically adjusts its S/I threshold when there is widely varying interference (e.g., ground clutter). Then the FAR stays constant.

This process is called Constant False Alarm Rate (CFAR), and it is used majority of modern radars.

Constant False Alarm Rate

- As was shown, the false alarm rate is very sensitive to the detection threshold voltage.
- Component aging and changes in background mean that a fixed detection threshold is not practical.
- Adaptive techniques that maintain a constant false alarm rate irrespective of the circumstances are called Constant False Alarm Rate (CFAR) processors.
- For aircraft this is not a problem as the area around the target is generally clear, and good background statistics can be obtained.
- For ground targets where the background is determined from clutter statistics, the terrain may not be homogeneous, and so additional processing is required.

Change the threshold to keep False Alarm Rate CONSTANT



Constant False Alarm Rate (CFAR)

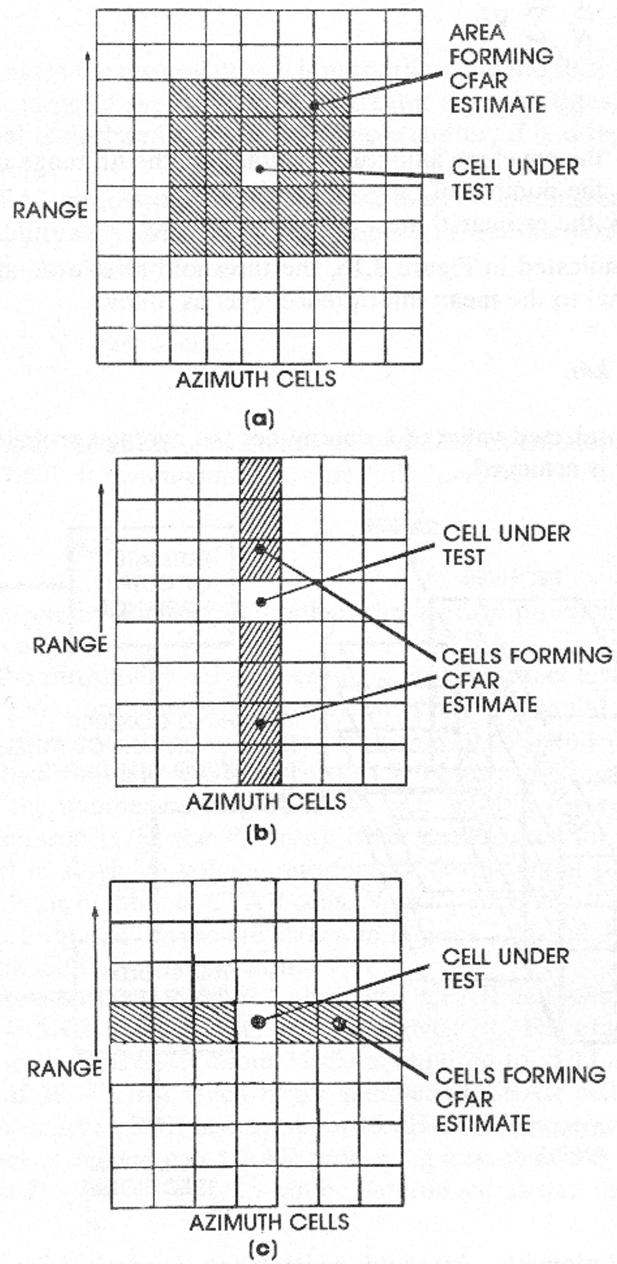
There are two common methods to achieve CFAR: Cell Averaging and Clutter Mapping.

Cell Averaging

When a running average is made of the range cells before and after the cell of interest. This running average is then multiplied by some factor and used as the threshold.

Cell Averaging CFAR Options

- Area CFAR used in imaging or scanning systems
- Range CFAR used by pencil beam radars
- Azimuth CFAR perimeter protection radar

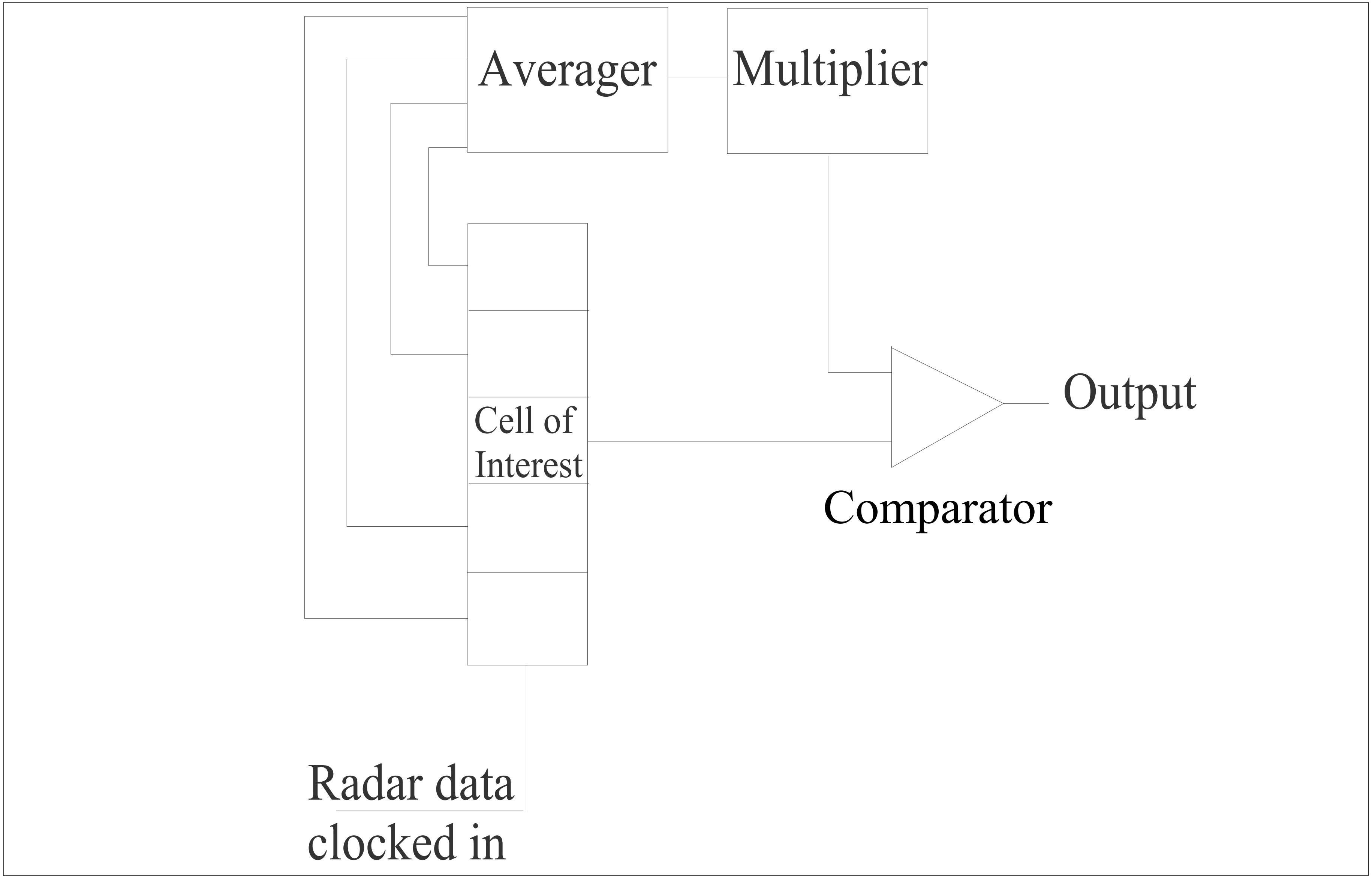


Range-azimuth cells used for (a) area CFAR (b) range-only CFAR, and (c) azimuth angle-only CFAR.

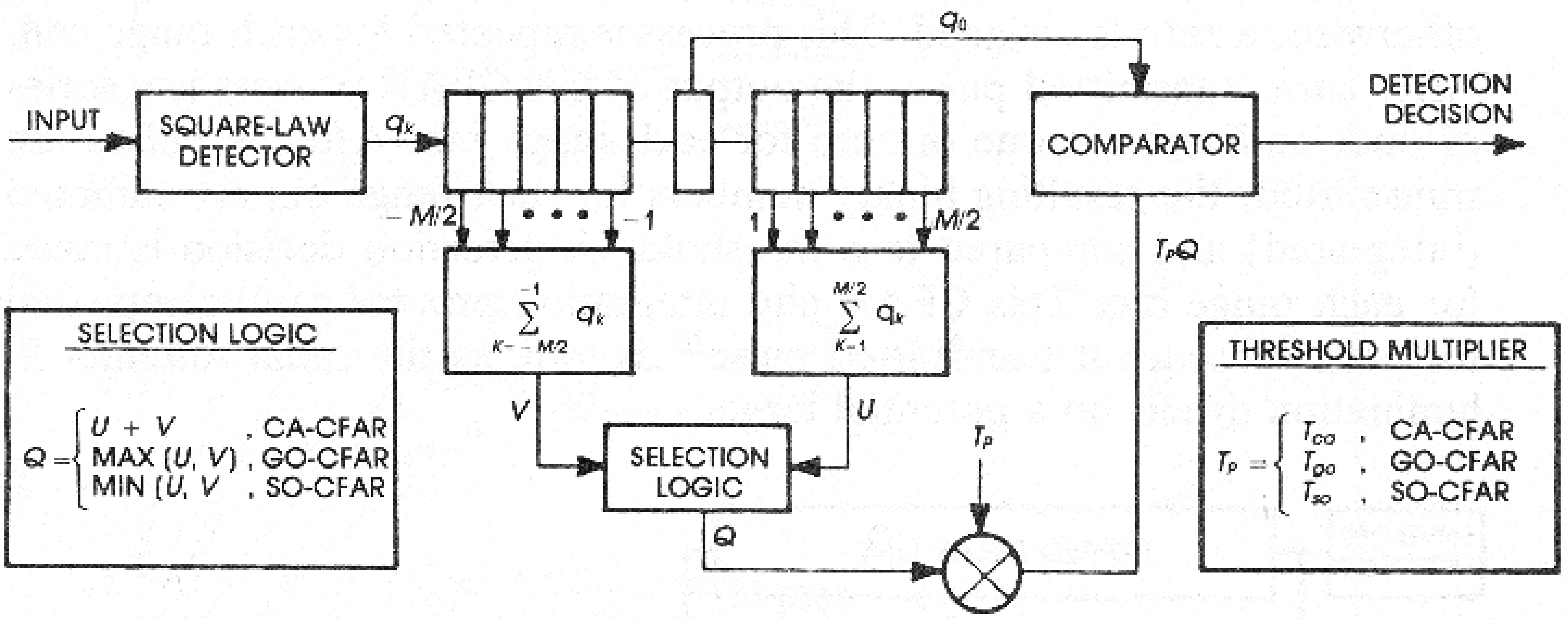
CFAR losses

- CFAR losses decrease with the number of cells used from 3.5dB for 10cells to 0.7dB for 40cells
- CFAR losses decrease with pulses integrated for a 10cell average with 10 pulses integrated it is 0.7dB decreasing to 0.3dB for 100 pulses

CFAR. Cell Averaging



Compensating for Non Homogeneous Clutter



CA=cell averaging
GO=Greater of
SO=Smallest of

Detection on the background of non-Gaussian interferences

- For example, stationary nonwhite noise. It is characterized by nonuniform spectrum $N(f)$.
- In this case instead MF we can introduce the notion of OPTIMUM filter, which takes into account not only characteristics of the signal, but characteristics of noise as well.
- Earlier: $N(f)=N_0=const$
- Now: $N(f)$ – is arbitrary function

Frequency response of the OPTIMAL FILTER

$$K_{opt}(f) = \frac{k_0 S_x^*(f) e^{-j2\pi fT}}{N(f)} \propto \frac{K_{MF}(f)}{N(f)}$$

Matched Filter is a special case of the Optimum Filter at $N(f)$=$const$

Physical interpretation of optimal filtering

- Optimal filter can be represented as a cascade connection of two “partial” filters:

$$\longrightarrow \boxed{K_1(f)} \longrightarrow \boxed{K_2(f)} \longrightarrow$$

- The first one makes the noise to become white (“whitewashing” filter), and the second one is matched with the signal, converted by the first filter.
- Optimal frequency response $K_1(f)K_2(f)$
- It is “whitewashing” noise, suppressing spectral components with great spectral densities of noise.

A priori uncertainty

- Two classes of the prior uncertainty:
  - **Parametric**

    A model of PDFs of signals $p_\mu(s)$ and interferences $p_\nu(n)$ are known; just parameters μ and ν are unknown (vectors)
  - **Nonparametric**

    Models of PDFs $p_\mu(s)$ and $p_\nu(n)$ are unknown.

    This is the most difficult case, but normally something is known (not complete uncertainty, but partial uncertainty)

Approaches to overcome a priori uncertainty

- Bayesian approach
  - Strictly Bayesian
  - Partially Bayesian
- Non-Bayesian parametric methods
- Non-parametric methods
  - Sign algorithms
  - Rank algorithms
- Adaptive optimal algorithms
- Adaptive non-optimal algorithms (Robust)

Bayesian approach

- Suppose that unknown parameters μ and ν

of PDFs $p_{\mu}(s)$ and $p_{\nu}(n)$ can be interpreted as random values, ar
their PDFs are exist.

For signal and noise $p_{\mu}(s)$ and $p_{\nu}(n)$ are considered as CONDITIONAL PDFs:

$$p_{\mu}(s/\mu) \qquad\qquad p_{\nu}(n/\nu)$$

$\mu = \{\mu_1, \mu_2, ...\}$ and $\nu = \{\nu_1, \nu_2, ...\}$ - vectors

- Then two statements of the problem are possible.
- <u>Strictly Bayesian</u>: PDFs $p\ (\mu)$ and $p\ (\nu)$ are known.

in this case we can write joint PDFs and PDFs for signal and noise:

$$p(s,\mu) = p(s/\mu)\cdot p(\mu)$$

$$p(n,\nu) = p(n/\nu)\cdot p(\nu)$$

$$p(s) = \int_{\Omega_1} p(s/\mu)p(\mu)d\mu$$

$$p(n) = \int_{\Omega_0} p(n/\nu)p(\nu)d\nu$$

- There are no unknown parameters anymore. We have come to the problem with completely known distributions. Now it is possible to synthesize optimal detection algorithm.
- There are no principal difficulties but computing difficulties can be very significant.

- Partially Bayesian: PDFs $p(\mu)$ and $p(\nu)$ are unknown.
- We can apply, for example Bayesian postulate that a priori distributions are uniform distributions.

$$p(\mu) = const, \quad \mu \in \Omega_1$$

$$p(\nu) = const, \quad \nu \in \Omega_0$$

- This supposition characterize the maximum prior uncertainty relative to parameters μ and ν .
- Minimax!

Non-Bayesian parametric methods

- These are all those parametric methods that do not require Bayesian suppositions relative to unknown parameters μ and ν of distributions for signals $p(s/\mu)$ and noise $p(n/\nu)$.
- One of the approaches consists in substitution into these distributions, instead of unknown parameters μ and ν , their estimates $\hat{\mu}$ and $\hat{\nu}$.
- The estimates should be obtained using data of observation.
- Thus, this approach leads to adaptive algorithms.

Non-parametric methods

- These class of methods is applied to overcome non-parametric prior uncertainty. The basis – non-parametric methods of math statistics.
- Concrete applications are related basically with using SIGN and RANK statistics, which have some invariant properties.
- Let us $X = (x_1, x_2, ..., x_N)$ is initial sequence of observed values.

- Sign Statistics is a vector:

$$signX = (sign\, x_1, sign\, x_2, ..., sign\, x_N)$$

$$sign\,\xi = \begin{cases} 1\ , & \xi > 0 \\ 0\ , & \xi = 0 \\ -1, & \xi < 0 \end{cases}$$

It appears that distribution of value *sign*x is invariant in respect to initial distribution of receiving signal X independently on $p(X)$

$$p\{sign\,\xi\} = \left(\frac{1}{2}\right)^N$$

Based on this property, the algorithms are designed that operate with sign functions instead of initial signals. Of course the quality is worse than in case of parametric uncertainty.

- Rank Statistics. In this case the observed values

$$X = (x_1, x_2, \ldots., x_N)$$

are arranged in the order of increasing.

$$x_i \geq x_j \quad i > j$$

Then instead of values X, we use the numbers of corresponding components in variational series (Ranks).

Invariant properties of rank statistics are even stronger than in previous case.

Robust approach

- Robust (stable) algorithms are build without evident estimating of non-informative μ and ν parameters.